THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

26 Nov 2015   11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		9
2		6
3		3
4		5
5		5
TOTAL:		28

The following trigonometric identities might be useful:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

/9

Problem 1: Evaluate the following. If an improper integral diverges, say so.

/3 (a) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

$$\begin{aligned} &= \lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx \\ &= \lim_{b\to\infty} \Big[-\tfrac{1}{2}e^{-x^2} \Big]_0^b \\ &= \lim_{b\to\infty} \big(\tfrac{1}{2} - \tfrac{1}{2}e^{-b^2}\big) = \tfrac{1}{2} - 0 = \boxed{\frac{1}{2}} \end{aligned}$$

/3 (b) $\displaystyle\int_3^4 \frac{dx}{(x-3)^{2/3}}$

$$\begin{aligned} &= \lim_{b\to 3^+} \int_b^4 (x-3)^{-2/3}\,dx \\ &= \lim_{b\to 3^+} \Big[3(x-3)^{1/3} \Big]_b^4 \\ &= 3\lim_{b\to 3^+} \Big[(4-3)^{1/3} - (b-3)^{1/3} \Big] = 3(1-0) = \boxed{3} \end{aligned}$$

/3 (c) $\displaystyle\int \frac{\sin^5 x}{\cos^2 x}\,dx$

Rewriting the integrand as

$$\frac{\sin^5 x}{\cos^2 x} = \frac{\sin^4 x}{\cos^2 x}\sin x = \frac{(\sin^2 x)^2}{\cos^2 x}\sin x = \frac{(1-\cos^2 x)^2}{\cos^2 x}\sin x$$

suggests the substitution $u = \cos x$; $du = -\sin x\,dx$:

$$\begin{aligned} \int \frac{\sin^5 x}{\cos^2 x}\,dx &= \int \frac{(1-u^2)^2}{u^2}\,(-du) \\ &= -\int \frac{1-2u^2+u^4}{u^2}\,du \\ &= -\int \big(u^{-2} - 2 + u^2\big)\,du \\ &= -\big(-u^{-1} - 2u + \tfrac{1}{3}u^3\big) + C \\ &= \boxed{\frac{1}{\cos x} + 2\cos x - \tfrac{1}{3}\cos^3 x + C} \end{aligned}$$

/6

Problem 2: Evaluate the following:

/3

(a) $\displaystyle\int_4^5 \frac{5x}{x^2-x-6}\,dx$

Use partial fractions:

$$\frac{5x}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$x=3 \implies 15 = 5A \implies A = 3$$
$$x=-2 \implies -10 = -5B \implies B = 2$$

So

$$\begin{aligned}\int_4^5 \frac{5x}{x^2-x-6}\,dx &= \int_4^5 \frac{3}{x-3} + \frac{2}{x+2}\,dx \\ &= \Big[3\ln|x-3| + 2\ln|x+2|\Big]_4^5 \\ &= \big(3\ln 2 + 2\ln 7\big) - \big(3\ln 1 + 2\ln 6\big) \\ &= \boxed{3\ln 2 + 2\ln 7 - 2\ln 6} = \ln\frac{2^3\cdot 7^2}{6^2} = \boxed{\ln\frac{98}{9}}\end{aligned}$$

/3

(b) $\displaystyle\int \frac{x^2+x+2}{x^2+1}\,dx$

Use polynomial long division to simplify, or just be clever and write

$$\frac{x^2+x+2}{x^2+1} = \frac{x^2+1+x+1}{x^2+1} = \frac{x^2+1}{x^2+1} + \frac{x+1}{x^2+1} = 1 + \frac{x}{x^2+1} + \frac{1}{x^2+1}$$

so that

$$\begin{aligned}\int \frac{x^2+x+2}{x^2+1}\,dx &= \int 1 + \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx \\ &= \int 1\,dx + \int \frac{1}{x^2+1}\,dx + \int \frac{x}{x^2+1}\,dx \\ &= x + \arctan x + \int \frac{1}{2}\cdot\frac{du}{u} \qquad (u = x^2+1;\ du = 2x\,dx) \\ &= \boxed{x + \arctan x + \tfrac{1}{2}\ln(x^2+1) + C}\end{aligned}$$

/3

Problem 3: Use Simpson's Rule with 4 subintervals to estimate the value of $\displaystyle\int_1^2 \ln x\,dx$.

For 4 subintervals of $[0,1]$ we have

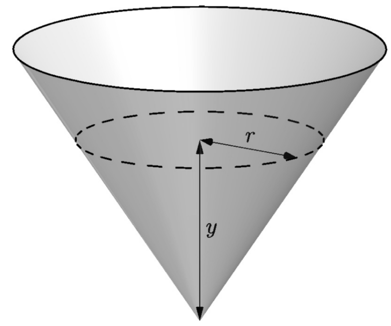
$$\Delta x = \frac{1}{4} = 0.25 \quad \text{and} \quad \begin{aligned} x_0 &= 1 \\ x_1 &= 1.25 \\ x_2 &= 1.5 \\ x_3 &= 1.75 \\ x_4 &= 2\end{aligned}$$

so

$$\begin{aligned}\int_1^2 \ln x\,dx &\approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &= \frac{0.25}{3}\left[\ln 1 + 4\ln 1.25 + 2\ln 1.5 + 4\ln 1.75 + \ln 2\right] \approx \boxed{0.386}\end{aligned}$$

/5

Problem 4: A water tank is shaped like an inverted cone with its vertex at the bottom, a radius of 2 m at the top, and height 6 m. If the tank is initially full of water (density $\rho = 1000\,\text{kg/m}^3$) how much work is required to pump all the water to the level of the top of the tank?



To remove a horizontal circular slice of thickness dy at height y from the bottom requires work

$$dW = mgh = \rho \underbrace{(\pi r^2 dy)}_{dV} g(6-y).$$

Similar triangles gives

$$\frac{r}{y} = \frac{2}{6} \implies r = \frac{y}{3}$$

so

$$dW = \pi\rho g \left(\frac{y}{3}\right)^2 (6-y)dy = \tfrac{1}{9}\pi\rho g y^2(6-y)dy.$$

Integrating gives the total work:

$$\begin{aligned} W = \int dW &= \tfrac{1}{9}\pi\rho g \int_0^6 y^2(6-y)\,dy = \tfrac{1}{9}\pi\rho g \int_0^6 (6y^2 - y^3)\,dy \\ &= \tfrac{1}{9}\pi\rho g \left[2y^3 - \tfrac{1}{4}y^4\right]_0^6 \\ &= \tfrac{1}{9}\pi\rho g(2\cdot 6^3 - \tfrac{1}{4}\cdot 6^4) = \boxed{12\pi\rho g \approx 3.7\times 10^5\ \text{J}} \end{aligned}$$

/5

Problem 5: Find the function $y(t)$ that satisfies: $\dfrac{dy}{dt} = y(4t^3+1), \quad y(0) = 4.$

Separating variables and integrating gives

$$\begin{aligned} \frac{dy}{y} = (4t^3+1)\,dt &\implies \ln|y| = t^4 + t + C \\ &\implies y = \pm e^{t^4+t+C} = Ae^{t^4+t}. \end{aligned}$$

This is the general solution of the given DE. Imposing the "initial condition" gives

$$y(0) = 4 = Ae^0 \implies A = 4$$

$$\implies \boxed{y = 4e^{t^4+t}}$$