THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

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MIDTERM EXAM #2
SOLUTIONS

13 Nov 2014   11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		15
2		9
3		8
4		8
TOTAL:		40

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Problem 1: Evaluate the following:

/4 (a) $\displaystyle\int_1^{\infty} \frac{1}{(3x+1)^2}\,dx$

$$= \lim_{b\to\infty} \int_1^b (3x+1)^{-2}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{3}(3x+1)^{-1}\right]_1^b$$

$$= -\frac{1}{3}\lim_{b\to\infty}\left[\frac{1}{3b+1} - \frac{1}{3(1)+1}\right] = \boxed{\frac{1}{12}}$$

/4 (b) $\displaystyle\int \frac{dx}{x^2+5x+6}$

partial fractions:

$$\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{A}{x+3} + \frac{B}{x+2} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

$$1 = A(x+3)+B(x+2) \quad : \quad \begin{cases} x=-2 & \implies 1 = A \\ x=-3 & \implies 1 = -B \implies B=-1 \end{cases}$$

Therefore

$$\int \frac{dx}{x^2+5x+6} = \int \frac{1}{x+2} - \frac{1}{x+3}\,dx = \boxed{\ln|x+2| - \ln|x+3| + C}$$

/4 (c) $\displaystyle\int \sin^5 x \cos^3 x\,dx$

$$\int \sin^5 x\cos^3 x\,dx = \int \sin^5 x\cos^2 x\cos x\,dx$$

$$= \int \sin^5 x(1-\sin^2 x)\cos x\,dx \qquad u = \sin x;\quad du = \cos x\,dx$$

$$= \int u^5(1-u^2)\,du$$

$$= \int u^5 - u^7\,du = \frac{1}{6}u^6 - \frac{1}{8}u^8 + C = \boxed{\frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C}$$

/3 (d) $f'(x)$ where $f(x) = \displaystyle\int_2^{3x} \frac{u^2-1}{u^2+1}\,du$

by the Fundamental Theorem of Calculus (with the chain rule):

$$\boxed{f'(x) = \frac{(3x)^2-1}{(3x)^2+1}\cdot 3}$$

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Problem 2: A certain random variable x has probability density function

$$f(x) = \begin{cases} kx(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) For what value of k is f a valid probability density function?

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$$\begin{aligned} 1 &= \int_{\infty}^{\infty} f(x)\,dx \\ &= \int_0^1 kx(1-x)\,dx \\ &= k\int_0^1 x - x^2\,dx \\ &= k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{k}{6} \implies \boxed{k = 6} \end{aligned}$$

(b) Find the probability that $x \ge \frac{1}{2}$.

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$$\begin{aligned} \text{Prob}(x \ge \frac{1}{2}) &= \int_{1/2}^{\infty} f(x)\,dx \\ &= \int_{1/2}^1 6x(1-x)\,dx \\ &= 6\int_{1/2}^1 x - x^2\,dx \\ &= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/2}^1 = 6\left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{8} - \frac{1}{24}\right)\right] = \boxed{\frac{1}{2}} \end{aligned}$$

(c) Find the expected (i.e. mean) value of x.

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$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} xf(x)\,dx \\ &= \int_0^1 x \cdot 6x(1-x)\,dx \\ &= 6\int_0^1 x^2 - x^3\,dx \\ &= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right] = \boxed{\frac{1}{2}} \end{aligned}$$

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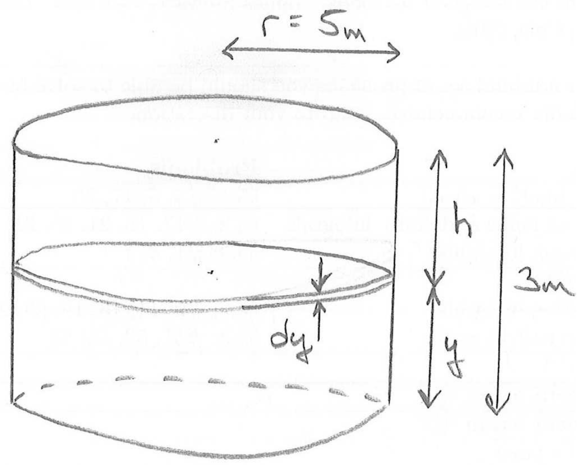
Problem 3: A circular swimming pool 10 m in diameter and 3 m deep contains water to a depth of 2 m. How much work is required to pump all the water out over the top edge of the pool?



Remove water to top of the pool by a sequence of circular "thin slices" as shown. Each slice requires work

$$dW = dm \cdot gh$$

where

$$h = 3 - y$$

and

$$dm = \rho \pi r^2 \, dy = \rho \pi (5)^2 \, dy = 25\pi\rho \, dy$$

where ρ is the density of water [kg/m^3]. So

$$dW = (25\pi\rho \, dy) g(3 - y)$$

and therefore the total work is

$$W = \int dW = \int_0^2 25\pi\rho g(3-y)\,dy$$
$$= 25\pi\rho g \int_0^2 (3-y)\,dy$$
$$= 25\pi\rho g \underbrace{\left[3y - \frac{y^2}{2}\right]_0^2}_{6-2=4} = \boxed{100\pi\rho g}$$

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Problem 4: A vessel with 2000 L of beer contains 4% alcohol (by volume). Beer with 8% alcohol is pumped into the vessel at a rate of 20 L/min and the mixture is pumped out at the same rate. What is the alcohol content (% by volume) after 1 hour?

Let $x(t)$ be the volume [L] of pure alcohol in the vessel after t minutes. Then

$$\begin{aligned}\frac{dx}{dt} &= \text{“rate in”} - \text{“rate out”} \\ &= (20\,\text{L/min})(0.08) - (20\,\text{L/min}) \cdot \frac{(x\,\text{L})}{(2000\,\text{L})} \\ &= 1.6 - 0.01x \quad [\text{L/min}]\end{aligned}$$

Solve this differential equation by separating variables:

$$\begin{aligned}\frac{dx}{1.6 - 0.01x} = dt &\implies -100 \ln|1.6 - 0.01x| = t + C \\ &\implies \ln|1.6 - 0.01x| = (t + C)/(-100) \\ &\implies |1.6 - 0.01x| = e^{(t+C)/(-100)} = Ae^{-0.01t} \\ &\implies 0.01x = 1.6 - Ae^{-0.01t} \\ &\implies x = 160 - Be^{-0.01t}\end{aligned}$$

Imposing the “initial conditions” gives

$$x(0) = (2000\,\text{L})(0.04) = 80\,\text{L} = 160 - Be^0 \implies B = 80$$

$$\implies x(t) = 160 - 80e^{-0.01t}$$

Thus after 1 h = 60 min the volume of pure alcohol in the vessel is

$$x(60) = 160 - 80e^{-0.01(60)} \approx 116.1\,\text{L}$$

so that the concentration [% by volume] is

$$\frac{116.1\,\text{L}}{2000\,\text{L}} \approx 0.058 = \boxed{5.8\%}$$