THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

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MIDTERM EXAM #2
SOLUTIONS

22 March 2013 09:30–10:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		12
2		6
3		7
4		8
TOTAL:		33

/12

Problem 1: Evaluate the following:

/4 (a) $$\int_0^\infty \frac{x}{1+2x^2}\,dx$$

$$\begin{array}{l} u = 1+2x^2 \\ du = 4x\,dx \end{array} \implies \int \frac{x}{1+2x^2}\,dx = \int \frac{1}{4}\frac{du}{u} = \frac{1}{4}\ln|u| = \frac{1}{4}\ln(1+2x^2)$$

$$\implies \int_0^\infty \frac{x}{1+2x^2}\,dx = \lim_{b\to\infty} \frac{1}{4}\ln(1+2x^2)\Big|_0^b$$

$$= \frac{1}{4}\lim_{b\to\infty}[\ln(1+2b^2) - 0] = \boxed{+\infty}$$

/4 (b) $$\int_0^2 \frac{dx}{4-x^2}$$

$$\int_0^2 \frac{dx}{4-x^2} = \lim_{b\to 2^-}\int_0^b \frac{A}{x+2} + \frac{B}{x-2}\,dx$$

$$= \lim_{b\to 2^-}\Big[A\ln|x+2| + B\ln|x-2|\Big]_0^b = \boxed{-\infty}$$

/4 (c) $$\int \frac{x^2+1}{6x-x^2}\,dx$$

Long division gives:

$$\frac{x^2+1}{6x-x^2} = -1 + \frac{6x+1}{6x-x^2}$$

and partial fractions gives

$$\frac{6x+1}{6x-x^2} = \frac{1/6}{x} + \frac{37/6}{6-x}$$

so we have

$$\int \frac{x^2+1}{6x-x^2}\,dx = \int -1 + \frac{1/6}{x} + \frac{37/6}{6-x}\,dx = \boxed{-x + \frac{1}{6}\ln|x| - \frac{37}{6}\ln|6-x| + C}$$

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Problem 2: Find the function $y(x)$ that satisfies the following differential equation and initial value:

$$\frac{dy}{dx} = 1 + y^2; \quad y(0) = 1.$$

Separate variables:

$$\int \frac{dy}{1+y^2} = \int dx \implies \arctan y = x + C$$

$$\implies y = \tan(x + C)$$

Impose initial conditions:

$$y(0) = 1 \implies 1 = \tan(0 + C) \implies C = \frac{\pi}{4}$$

$$\implies \boxed{y = \tan\left(x + \frac{\pi}{4}\right)}$$

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Problem 3: A spherical tank of radius 5 m is full of water (density 1000 kg/m^3).

(a) Calculate the work required to pump all the water out of the tank through a hole at the top. Express your answer as a definite integral, but do not evaluate this integral. (Use $g = 9.8\,\text{m/s}^2$).

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Viewed from the side, the boundary of the tank is a circle with equation

$$x^2 + y^2 = 25 \implies x^2 = 25 - y^2.$$

Consider a horizontal (circular) slice of water at y, with infinitesimal thickness dy. The volume of this slice is

$$dV = \pi r^2\,dy = \pi(25 - y^2)\,dy.$$

To pump this slice to the top of the tank (at $y = 5$) we need to raise it by $5 - y$, which requires work

$$\begin{aligned} dW &= \rho g\,dV(5-y) \\ &= \rho g \pi (25 - y^2)(5 - y)\,dy \end{aligned}$$

$$\implies W = \int_{-5}^{5} \rho g \pi (25 - y^2)(5-y)\,dy = \boxed{\rho g \pi \int_{-5}^{5} \underbrace{(25-y^2)(5-y)}_{f(y)}, dy}$$

(b) Use Simpson's Rule (with $n = 4$) to approximate the definite integral in part (a).

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We have $\Delta y = \frac{10}{4} = 2.5$.

$$\begin{aligned} \implies \int_{-5}^{5} f(y)\,dy &\approx \frac{\Delta y}{3}[f(-5) + 4f(-2.5) + 2f(0) + 4f(2.5) + f(5)] \\ &\approx \frac{2.5}{3}[0 + 4(140.6) + 2(125) + 4(46.9) + 0] \approx 833.3 \end{aligned}$$

So

$$W = \rho g \pi \int_{-5}^{5} f(y), dy \approx (1000)(9.8)(3.14)(833.3) \approx \boxed{25.7 \times 10^6\ \text{J}}$$

/8 **Problem 4:** For a certain brand of light bulb, let T be the time (in years after installation) at which any individual bulb burns out. T is a continuous random variable with probability density

$$f(t) = \begin{cases} Ce^{-t/2} & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of C such that $f(t)$ is a valid probability density function.

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We require

$$1 = \int_0^{\infty} f(t)\,dt = \lim_{b\to\infty} C\Big[-2e^{-t/2}\Big]_0^b = 2C \implies \boxed{C = \frac{1}{2}}$$

(b) Calculate the probability that a given bulb will last at least 5 years after it is installed.

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$$\begin{aligned}\int_5^{\infty} f(t)\,dt &= \int_5^{\infty} \frac{1}{2}e^{-t/2}\,dt \\ &= \lim_{b\to\infty} \frac{1}{2}\Big[-2e^{-t/2}\Big]_5^b = \boxed{e^{-5/2} \approx 0.082}\end{aligned}$$

(c) Find the average time that this type of bulb with last before burning out.

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$$\begin{aligned}\bar{t} = \int_0^{\infty} tf(t)\,dt &= \int_0^{\infty} t \cdot \frac{1}{2}e^{-t/2}\,dt \quad \text{(integrate by parts)} \\ &= \lim_{b\to\infty} -(t+2)e^{-t/2}\Big|_0^b = \boxed{2\text{ years}}\end{aligned}$$