THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus 2

Instructor: Richard Taylor

MIDTERM EXAM #1
SOLUTIONS

14 Feb. 2019 08:30–09:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		13
2		9
3		5
4		6
TOTAL:		33

/13

Problem 1: Evaluate the following:

(a) $\displaystyle\int \left(10x^2 + \frac{8}{x} - 2e^3 + \frac{1}{3\sqrt{x}}\right) dx$

/3

$$\int \left(10x^2 + \frac{8}{x} - 2e^3 + \frac{1}{3}x^{-1/2}\right) dx = \boxed{\tfrac{10}{3}x^3 + 8\ln|x| - 2e^3 x + \tfrac{2}{3}x^{1/2} + C}$$

(b) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$

/4

Substitute: $u = 1 + \sqrt{x} = 1 + x^{1/2} \implies du = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$:

$$\int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})^2} = \int_2^3 \frac{2\,du}{u^2} = \int_2^3 2u^{-2}\,du = -2u^{-1}\Big|_2^3 = \frac{2}{2} - \frac{2}{3} = \boxed{\frac{1}{3}}$$

(c) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

/3

Substitute: $u = \ln x \implies du = \dfrac{1}{x}\,dx$:

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \boxed{\tfrac{1}{3}(\ln x)^3 + C}$$

(d) $\displaystyle\frac{d}{dx}\int_{\sqrt{x}}^0 \sin(t^2)\,dt$

/3

We have

$$\int_{\sqrt{x}}^0 \sin(t^2)\,dt = g(x) = f(\sqrt{x})$$

where

$$f(x) = \int_x^0 \sin(t^2)\,dt = -\int_0^x \sin(t^2)\,dt \implies f'(x) = -\sin(x^2).$$

So by the chain rule:

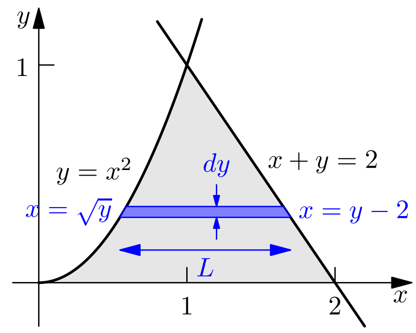
$$g'(x) = f'(\sqrt{x})\frac{d}{dx}\sqrt{x} = f'(\sqrt{x}) \cdot \tfrac{1}{2}x^{-1/2} = \boxed{-\sin\left((\sqrt{x})^2\right) \cdot \tfrac{1}{2}x^{-1/2} = -\frac{\sin x}{2\sqrt{x}}}$$

/9

Problem 2: Consider the shaded region in the graph below.

/3

(a) Calculate the area of the shaded region.



By horizontal strips as shown:

$$A = L\,dy = [(2-y) - \sqrt{y}]\,dy$$

$$\implies A = \int dA = \int_0^1 (2 - y - y^{1/2})\,dy$$

$$= 2y - \tfrac{1}{2}y^2 - \tfrac{2}{3}y^{3/2}\Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \boxed{\frac{5}{6}}$$

(b) Write (but do not evaluate) a definite integral that represents the volume of the solid formed by revolving the shaded region about the x-axis.

/3

By "thin shells" (revolving each horizontal strip about the x-axis):

$$dV = (\text{circumference})(L)\,dy = (2\pi y)(L)\,dy = 2\pi y(2 - y - \sqrt{y})\,dy$$

$$\implies V = \int dV = \boxed{\int_0^1 2\pi y(2 - y - \sqrt{y})\,dy = 2\pi \int_0^1 (2y - y^2 - y^{3/2})\,dy}$$

(c) Write (but do not evaluate) a definite integral that represents the volume of the solid formed by revolving the shaded region about the y-axis.

/3

By "thin discs" (revolving each horizontal strip about the y-axis):

$$dV = (\pi R^2 - \pi r^2)\,dy = \big(\pi(2-y)^2 - \pi(\sqrt{y})^2\big)\,dy$$

$$\implies V = \int dV = \boxed{\int_0^1 \big(\pi(2-y)^2 - \pi(\sqrt{y})^2\big)\,dy = \pi \int_0^1 \big((2-y)^2 - y\big)\,dy}$$

/5 **Problem 3:** The population of a community of foxes is observed to have a growth rate

$$P'(t) = 5 + 10\sin\frac{\pi t}{5} \quad \text{[foxes/yr]}$$

with t measured in years. At $t = 0$ the population was 35 foxes.

(a) Interpret the physical meaning of (but do not evaluate) the quantity $\int_1^2 P'(t)\,dt$.

/2

net change of population in the month from $t = 1$ to $t = 2$

(b) Calculate the population of foxes at $t = 5$.

/3

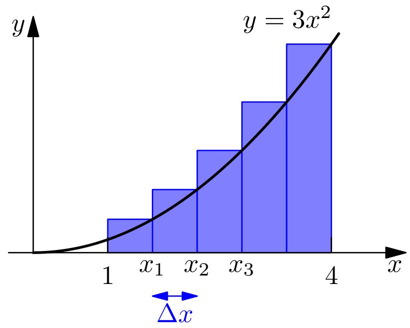
$$\begin{aligned}
P(5) &= P(0) + \int_0^5 P'(t)\,dt \\
&= 35 + \int_0^5 \left(5 + 10\sin\frac{\pi t}{5}\right) dt \\
&= 35 + \int_0^5 5\,dt + 10\int_0^5 \sin\frac{\pi t}{5}\,dt \\
&= 35 + 25 + 10\int_0^{\pi} \sin(u)\frac{5}{\pi}du \quad \left(u = \frac{\pi t}{5};\ \ du = \frac{\pi}{5}dt\right) \\
&= 35 + 25 + \frac{50}{\pi}\big[-\cos u\big]_0^{\pi} \\
&= 35 + 25 + \frac{50}{\pi}\cdot 2 \\
&= \boxed{60 + \frac{100}{\pi} \approx 91.8 \approx 92 \text{ foxes}}
\end{aligned}$$

/6

Problem 4: Use the definition of the definite integral (i.e. as a limit of Riemann sums) to evaluate:

$$\int_1^4 3x^2\,dx$$

The following formulas might be useful: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$



With n rectangles:

$$\Delta x = \frac{3}{n}$$
$$x_i = 1 + i\Delta x = 1 + \frac{3i}{n} \quad (i = 1, 2, \ldots, n)$$

$$\begin{aligned}
\int_1^4 3x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 3x_i^2 \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 3\left(1 + \frac{3i}{n}\right)^2 \frac{3}{n} \\
&= \lim_{n\to\infty} \frac{9}{n} \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{9}{n} \sum_{i=1}^{n} \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) \\
&= \lim_{n\to\infty} \frac{9}{n} \left(\sum_{i=1}^{n} 1 + \frac{6}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \frac{9}{n} \left(n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right) \\
&= 9 + 27 \underbrace{\lim_{n\to\infty} \frac{n(n+1)}{n^2}}_{1} + \frac{81}{6} \underbrace{\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{n^2}}_{2} \\
&= 9 + 27 + 27 = \boxed{63}
\end{aligned}$$