THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

Instructor: Richard Taylor

MIDTERM EXAM #1
SOLUTIONS

11 Feb 2016 10:00–11:15

PROBLEM	GRADE	OUT OF
1		12
2		3
3		6
4		3
5		6
6		6
7		6
TOTAL:		42

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

/12

Problem 1: Evaluate the following:

/3 (a) $\displaystyle\int \left(x + \frac{2}{x} - \frac{2}{x^2} + e^{-x} + \pi^2\right) dx$

Integrate term by term:

$$\int \left(x + \frac{2}{x} - \frac{2}{x^2} + e^{-x} + \pi^2\right) dx = \boxed{\frac{1}{2}x^2 + 2\ln|x| + \frac{2}{x} - e^{-x} + \pi^2 x + C}$$

/3 (b) $\displaystyle\int_1^{16} \frac{x + \sqrt{x}}{x^4}\, dx$

Simplify before integrating:

$$\int_1^{16} \frac{x+\sqrt{x}}{x^4}\, dx = \int_1^{16} (x^{-3} + x^{-7/2})\, dx = \left[-\frac{1}{2}x^{-2} - \frac{2}{5}x^{-5/2}\right]_1^{16}$$
$$= \left[-\frac{1}{2}\cdot\frac{1}{256} - \frac{2}{5}\cdot\frac{1}{1024}\right] - \left[-\frac{1}{2} - \frac{2}{5}\right] = \boxed{\frac{1149}{1280} \approx 0.898}$$

/3 (c) $\displaystyle\int_0^1 xe^{-x^2}\, dx$

Substitution: $u = x^2 \implies du = 2x\, dx$

$$\implies \int_0^1 xe^{-x^2}\, dx = \int_0^1 \tfrac{1}{2}e^{-u}\, du = -\tfrac{1}{2}e^{-u}\Big|_0^1 = \boxed{\tfrac{1}{2}(1 - e^{-1}) \approx 0.316}$$

/3 (d) $\displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\, dx$

Integrate by parts to find an anti-derivative:

$$u = (\ln x)^2 \qquad dv = x^{-3}\, dx$$
$$du = 2\ln x \cdot \frac{1}{x}\, dx \qquad v = -\frac{1}{2}x^{-2}$$

$$\implies \int \frac{(\ln x)^2}{x^3}\, dx = -\frac{1}{2}x^{-2}(\ln x)^2 + \int x^{-3}\ln x\, dx$$

By parts again:

$$u = \ln x \qquad dv = x^{-3}\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = -\frac{1}{2}x^{-2}$$

$$\implies \int x^{-3}\ln x\, dx = -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}\, dx$$
$$= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2}$$

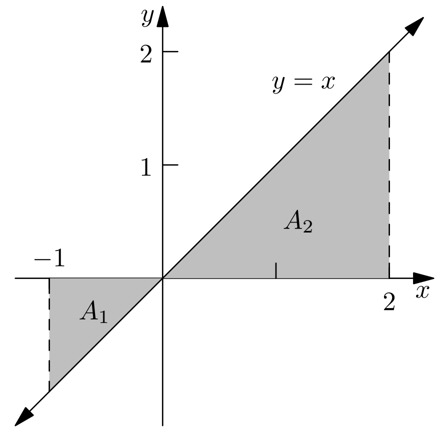
So finally:

$$\int_1^2 \frac{(\ln x)^2}{x^3}\, dx = \left[-\frac{1}{2x^2}(\ln x)^2 - \frac{1}{2x^2}\ln x - \frac{1}{4x^2}\right]_1^2 = \boxed{\frac{-(\ln 2)^2}{8} - \frac{\ln 2}{8} + \frac{3}{16} \approx 0.0408}$$

/3 **Problem 2:** Evaluate $\int_{-1}^{2} x\,dx$ by interpreting the integral in terms of areas of triangles. *Do not use an antiderivative.*

$$\int_{-1}^{2} x\,dx = -A_1 + A_2$$

$$= -\tfrac{1}{2}(1)(2) + \tfrac{1}{2}(2)(2) = \boxed{\frac{3}{2}}$$



/6 **Problem 3:** Find the area bounded between the graphs of $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$. (These curves intersect at the points $(-1,-2)$ and $(5,4)$).

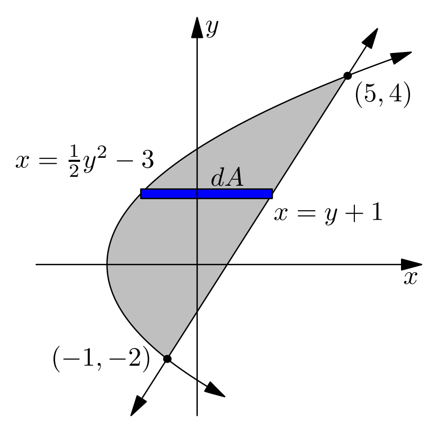
By "horizontal slices":

$$dA = (x_2 - x_1)\,dy = \left[(y+1) - (\tfrac{1}{2}y^2 - 3)\right]dy$$

$$= \left[y - \tfrac{1}{2}y^2 + 4\right]dy$$

$$\implies A = \int dA = \int_{-2}^{4} (y - \tfrac{1}{2}y^2 + 4)\,dy$$

$$= \left[\tfrac{1}{2}y^2 - \tfrac{1}{6}y^3 + 4y\right]_{-2}^{4} = \boxed{18}$$



/3 **Problem 4:** Find the average value of $y = \dfrac{1}{1+x^2}$ on the interval $[0, \frac{\pi}{4}]$.

Using the formula for the average value of a function on the interval $[0, \pi/4]$:

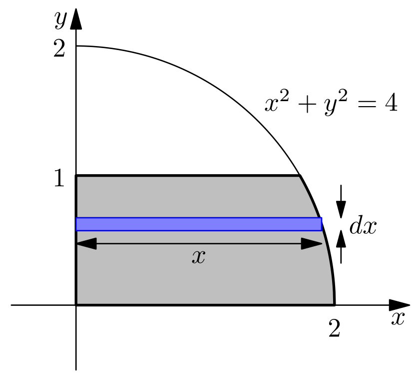
$$\bar{y} = \frac{1}{\pi/4}\int_0^{\pi/4} \frac{1}{1+x^2}\,dx$$

$$= \frac{4}{\pi}\frac{1}{1+x^2}\,dx$$

$$= \frac{4}{\pi}\arctan x\Big|_0^{\pi/4}$$

$$= \frac{4}{\pi}\left(\arctan(\tfrac{\pi}{4}) - \arctan 0\right) = \boxed{\frac{4}{\pi}\arctan\frac{\pi}{4} \approx 0.848}$$

/6 **Problem 5:** A solid object is formed by revolving the shaded region about the y-axis. Find the volume of this object.



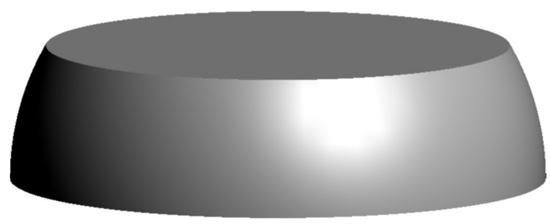
By horizontal slices ("discs"):

$$dV = \pi x^2\,dy = \pi(4 - y^2)\,dy$$

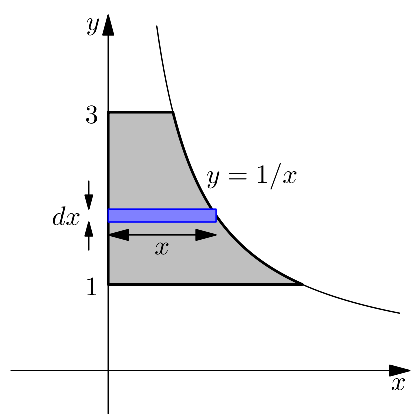
$$\Longrightarrow\ V = \int dV = \int_0^1 \pi(4 - y^2)\,dy$$

$$= \pi\left[4y - \tfrac{1}{3}y^3\right]_0^1 = \boxed{\frac{11\pi}{3}}$$

The object is a half-sphere with part of its top removed:



/6 **Problem 6:** A solid object is formed by revolving the shaded region about the x-axis. Find the volume of this object.

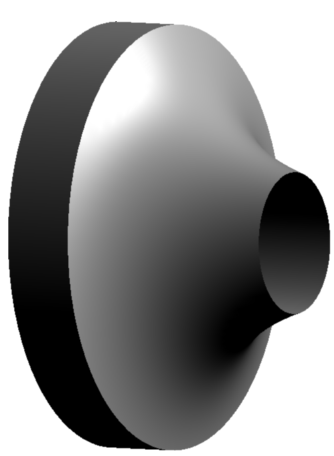


By horizontal slices ("cylindrical shells"):

$$dV = (2\pi y)x\,dy = 2\pi y \cdot \frac{1}{y}\,dy = 2\pi\,dy$$

$$\Longrightarrow\ V = \int dV = \int_1^3 2\pi\,dy$$

$$= 2\pi y\Big|_1^3 = \boxed{4\pi}$$



/6 **Problem 7:** Evaluate $\int_0^3 (2x-1)\,dx$ using the definition of the definite integral (i.e. by a limit of Riemann sums).

The following formulas may be useful: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

First construct the Riemann sum (area using n rectangles based on right endpoints):

$$\Delta x = \frac{3}{n} \qquad x_i = i\,\Delta x = \frac{3i}{n}$$

This gives for the area of n rectangles:

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\,\Delta x &= \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3i}{n} \\
&= \sum_{i=1}^{n} \left(2\cdot\frac{3i}{n} - 1\right)\frac{3}{n} \\
&= \sum_{i=1}^{n} \left(\frac{18i}{n^2} - \frac{3}{n}\right) \\
&= \frac{18}{n^2}\sum_{i=1}^{n} i - \frac{3}{n}\sum_{i=1}^{n} 1 \\
&= \frac{18}{n^2}\cdot\frac{n(n+1)}{2} - \frac{3}{n}\cdot n \\
&= \frac{9(n+1)}{n} - 3
\end{aligned}$$

Now let $n \to \infty$:

$$\int_0^3 (2x-1)\,dx = \lim_{n\to\infty} \frac{9(n+1)}{n} - 3 = 9 - 3 = \boxed{6}$$