THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

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MIDTERM EXAM #1
SOLUTIONS

6 February 2014 10:00–11:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		16
2		8
3		8
4		8
5		8
TOTAL:		48

/16 **Problem 1:** Evaluate the following:

/4 (a) $\displaystyle\int \frac{x^4-8}{x^2} + \frac{3}{x-5}\,dx$

$$= \int x^2 - 8x^{-2} + \frac{3}{x-5}\,dx$$

$$= \boxed{\frac{1}{3}x^3 + \frac{8}{x} + 3\ln|x-5| + C}$$

/4 (b) $\displaystyle\int_0^{\pi/2} \sin x \sin(\cos x)\,dx$

$$\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \end{aligned} \implies \int_1^0 -\sin u\,du = \int_0^1 \sin u\,du = -\cos u\Big|_0^1$$

$$= \boxed{1-\cos 1}$$

/4 (c) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2} + (1+x^2+3x^8)\sin x\,dx$

$$= \underbrace{\int_{-1}^{1}\sqrt{1-x^2}\,dx}_{=\frac{\pi}{2}\text{(area of semi-circle)}} + \underbrace{\int_{-1}^{1}(1+x^2+3x^8)\sin x\,dx}_{=0\text{(odd function on symmetric interval)}} = \boxed{\frac{\pi}{2}}$$

/4 (d) $\int x^5 \ln x^7\,dx$

$$\begin{aligned} u &= \ln x^7; & dv &= x^5\,dx \\ du &= \tfrac{7}{x}\,dx; & v &= \tfrac{x^6}{6} \end{aligned} \implies \int x^5 \ln x^7\,dx = \frac{x^6}{6}\ln x^7 - \int \frac{x^6}{6}\cdot\frac{7}{x}\,dx$$

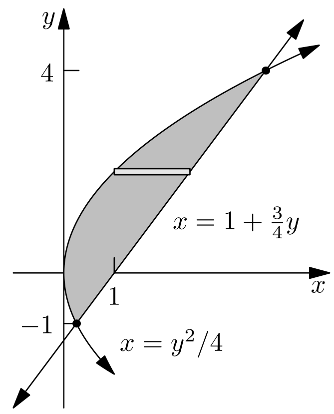
$$= \frac{1}{6}x^6\ln x^7 - \frac{7}{6}\int x^5\,dx$$

$$= \boxed{\frac{1}{6}x^6\ln x^7 - \frac{7}{36}x^6 + C}$$

/8 **Problem 2:** Sketch and find the area of the region bounded between the parabola $y^2 = 4x$ and the line $4x - 3y = 4$.

Note the following:

$$4x - 3y = 4 \implies x = 1 + \frac{3}{4}y \quad \text{and} \quad y^2 = 4x \implies x = \frac{1}{4}y^2.$$



Solve for intersection points:

$$x = 1 + \frac{3}{4}y = \frac{1}{4}y \implies y^2 - 3y - 4 = 0$$
$$\implies (y-4)(y+1) = 0 \implies y = -1,\, 4.$$

For a given horizontal strip of infinitesimal height dy:

$$dA = \left[(1 + \frac{3}{4}y) - \frac{1}{4}y^2)\right]\, dy$$

so

$$A = \int dA = \int_{-1}^{4} 1 + \frac{3}{4}y - \frac{1}{4}y^2\, dy$$
$$= \left[y + \frac{3}{8}y^2 - \frac{1}{12}y^3\right]_{-1}^{4} = \boxed{\frac{125}{4}}$$

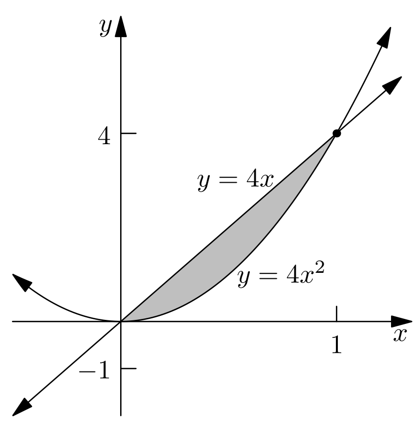
/8 **Problem 3:** Find the volume of the solid generated by revolving about the y-axis the region bounded by the line $y = 4x$ and the parabola $y = 4x^2$.

By cylindrical shells:

Revolving an vertical slize (infinitesimal width x) about $y = 0$ creates a cylindrical shell that contributes volume

$$dV = 2\pi rh\, dx = 2\pi x(4x - 4x^2)\, dx = 8\pi(x^2 - x^3)\, dx$$

$$\implies V = \int dV = \int_0^1 8\pi(x^2 - x^3)\, dx$$
$$= 8\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \boxed{\frac{2\pi}{3}}$$



By discs/washers:

Note that $y = 4x \implies x = \frac{y}{4} \implies r^2 = x^2 = \frac{y^2}{16}$ and also $y = 4x^2 \implies R^2 = x^2 = \frac{y}{4}$.

Revolving a horizontal slice (infinitesimal height dy) about $y = 0$ creates a "washer" that contributes volume

$$dV = \pi(R^2 - r^2)\, dy = \pi\left(\frac{y}{4} - \frac{y^2}{16}\right)$$

$$\implies V = \int dV = \pi\int_0^4 \frac{y}{4} - \frac{y^2}{16}\, dy = \pi\left[\frac{y^2}{8} - \frac{y^3}{48}\right]_0^4 = \boxed{\frac{2\pi}{3}}$$

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Problem 4: Sketch and find the area of the region under the curve $y = \frac{1}{2}x^2 + 1$ over the interval $[0, 1]$. To do this first find a formula for S_n (the approximate area as given by the Riemann sum of areas of n rectangles of equal width) then let $n \to \infty$.

The following formulas might be useful: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$$\Delta x = \frac{1}{n} \qquad x_i = i\Delta x = \frac{i}{n}$$

$$\begin{aligned} \implies S_n = \sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left[\frac{1}{2}\left(\frac{i}{n}\right)^2 + 1\right]\frac{1}{n} \\ &= \frac{1}{2n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 1 \\ &= \frac{1}{2n^3} \cdot \frac{n(n+1)(2n+1)}{6} + 1 \\ &= \frac{2n^3 + 3n^2 + n}{12n^3} + 1 \end{aligned}$$

$$\begin{aligned} \implies \int_0^1 f(x)\,dx = \lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(\frac{2 + 3/n + 1/n^2}{12} + 1\right) \\ &= \frac{2}{12} + 1 = \boxed{\frac{7}{6}} \end{aligned}$$

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Problem 5: The figure below shows the graph of $y = v(t)$ where v is the velocity of an object moving in one dimension (v is measured in units of meters per second; t is measured in seconds).

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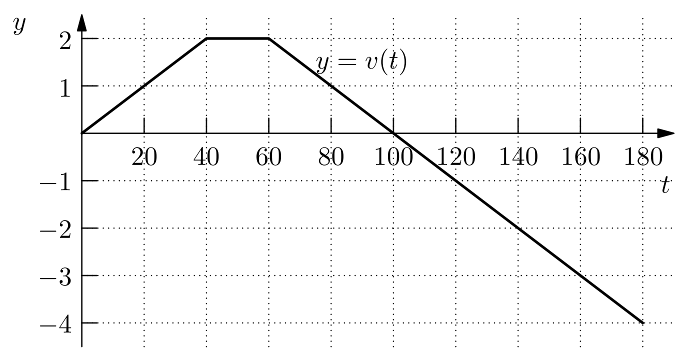
(a) Evaluate $\int_{40}^{120} v(t)\,dt$.



We can do this geometrically by summing areas of rectangle and triangles:

$$\int_{40}^{120} v(t)\,dt = (20)(2) + \frac{1}{2}(40)(2) - \frac{1}{2}(20)(1) = \boxed{70\,\text{m}}$$

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(b) What physical interpretation can one give to the quantity $\int_{40}^{120} v(t)\,dt$?

The integral represents the net displacement (change in position) from $t = 40$ to $t = 120$.

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(c) At what time (if any) will the object return to its initial position (i.e. at $t = 0$)?

We're looking for the time T at which $\int_0^T v(t)\,dt = 0$. Again, by summing areas of triangles and rectangles this integral evaluates to

$$\int_0^T v(t)\,dt = \frac{1}{2}(40)(2) + (20)(2) + \frac{1}{2}(40)(2) - \frac{1}{2}(T - 100)\frac{T - 100}{20} = 0.$$

$$\implies 120 - \frac{(T-100)^2}{40} = 0 \implies T = 100 + \sqrt{4800} = \boxed{100 + 40\sqrt{3} \approx 169.3\,\text{s}}$$