THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

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MIDTERM EXAM #1
SOLUTIONS

13 October 2016 11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		12
2		5
3		5
4		8
5		8
TOTAL:		38

/12

Problem 1: Evaluate the following:

/3 (a) $\displaystyle\int_4^9 \frac{2+\sqrt{x}}{x}\,dx$

$$\begin{aligned}\int_4^9 \frac{2+\sqrt{x}}{x}\,dx &= \int_4^9 \frac{2}{x}+\frac{\sqrt{x}}{x}\,dx \\ &= \int_4^9 \frac{2}{x}+x^{-1/2}\,dx \\ &= \Big[2\ln|x|+2x^{1/2}\Big]_4^9 \\ &= \Big[2\ln|9|+2\cdot 3\Big]-\Big[2\ln|4|+2\cdot 2\Big] = \boxed{2\ln 9 - 2\ln 4 + 2 = 2\ln\frac{9}{4}+2}\end{aligned}$$

/3 (b) $\displaystyle\int_0^{\pi/4} \cos 2x\,dx$

$$\begin{aligned}\int_0^{\pi/4}\cos 2x\,dx &= \frac{1}{2}\sin 2x\Big|_0^{\pi/4} \\ &= \frac{1}{2}\sin\frac{\pi}{2}-\frac{1}{2}\sin 0 = \boxed{\frac{1}{2}}\end{aligned}$$

/3 (c) $\displaystyle\int \frac{2x^2}{\sqrt{1-4x^3}}\,dx$

Substitute:

$$\begin{cases} u = 1-4x^3 \\ du = -12x^2\,dx\end{cases} \implies \int \frac{2x^2}{\sqrt{1-4x^3}}\,dx = \int \frac{-du/6}{\sqrt{u}}$$

$$\begin{aligned} &= -\frac{1}{6}\int u^{-1/2}\,du \\ &= -\frac{1}{6}\cdot 2u^{1/2}+C \\ &= \boxed{-\frac{1}{3}\sqrt{1-4x^3}+C}\end{aligned}$$

/3 (d) $\displaystyle\int \frac{\ln x}{x^{10}}\,dx$

Integrate by parts:

$$u = \ln x \qquad dv = x^{-10}\,dx$$

$$du = \frac{1}{x}\,dx \quad v = -\frac{1}{9}x^{-9}$$

$$\begin{aligned}\int \frac{\ln x}{x^{10}}\,dx = \int u\,dv &= uv - \int v\,du \\ &= -\frac{1}{9}x^{-9}\ln x - \int\left(-\frac{1}{9}x^{-9}\right)\frac{1}{x}\,dx \\ &= -\frac{1}{9}x^{-9}\ln x + \frac{1}{9}\int x^{-10}\,dx \\ &= \boxed{-\frac{1}{9}x^{-9}\ln x - \frac{1}{81}x^{-9}+C}\end{aligned}$$

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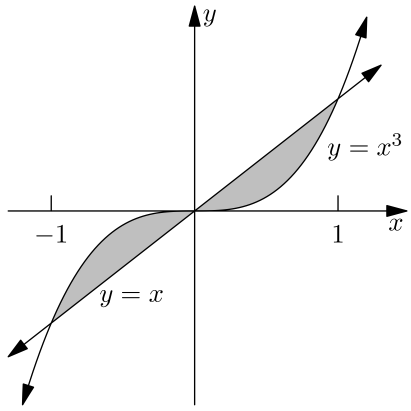
Problem 2: Sketch and find the area bounded between the graphs of $y = x$ and $y = x^3$.



$$\begin{aligned} A &= \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx \\ &= 2\int_0^1 (x - x^3)\,dx \quad \text{(by symmetry)} \\ &= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 \\ &= 2\left[\frac{1}{2} - \frac{1}{4}\right] = \boxed{\frac{1}{2}} \end{aligned}$$

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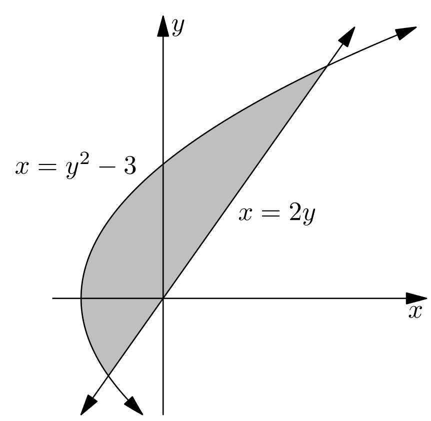
Problem 3: Find the area of the shaded region.



Intersection points:

$$y^2 - 3 = 2y \implies \underbrace{y^2 - 2y - 3}_{(y-3)(y+1)} = 0 \implies y = -1 \text{ or } 3.$$

By horizontal strips:

$$dA = \left[2y - (y^2 - 3)\right] dy = \left[2y - y^2 + 3\right] dy$$

$$\begin{aligned} \implies A = \int dA &= \int_{-1}^{3} \left[2y - y^2 + 3\right] dy \\ &= \left[y^2 - \frac{1}{3}y^3 + 3y\right]_{-1}^{3} \\ &= [9 - 9 + 9] - [1 + \frac{1}{3} - 3] = \boxed{\frac{32}{3}} \end{aligned}$$

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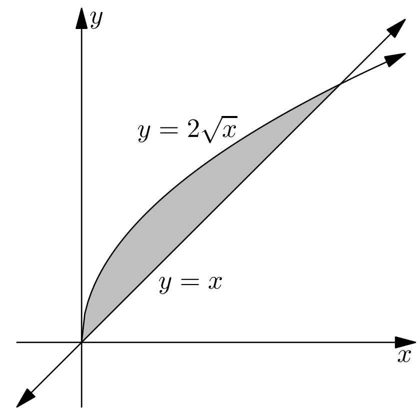
Problem 4: A solid of revolution is formed by revolving, about the x-axis, the region bounded by the graphs of $y = x$ and $y = 2\sqrt{x}$. Calculate the volume of this solid.



Using the method of washers (thin slices perpendicular to the x-axis):

$$\begin{aligned} dV &= \pi(R^2 - r^2)\,dx \quad \text{where} \quad \begin{cases} R = 2\sqrt{x} \\ r = x \end{cases} \\ &= \pi\left((2\sqrt{x})^2 - (x)^2\right) \\ &= \pi(4x - x^2) \end{aligned}$$

To get the limits of integration we need the intersection points:

$$x = 2\sqrt{x} \implies x^2 = 4x \implies x(x-4) = 0 \implies x = 0 \text{ or } 4$$

$$\begin{aligned} V &= \int dV \\ &= \int_0^4 \pi(4x - x^2)\,dx \\ &= \pi\left[2x^2 - \frac{1}{3}x^3\right]_0^4 \\ &= \pi\left[2 \cdot 4^2 - \frac{1}{3} \cdot 4^3\right] = \boxed{\frac{32\pi}{3}} \end{aligned}$$

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Problem 5: Use Riemann sums to evaluate the definite integral:

$$\int_1^3 x^2\,dx$$

The following formulas might be useful: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

The right-Riemann sum (area of n rectangles) is

$$A = \sum_{i=1}^{n} f(x_i)\Delta x$$

where for n equal rectangles in the interval $[1,3]$ we have

$$\Delta x = \frac{2}{n}, \qquad x_i = 1 + i\Delta x = 1 + \frac{2i}{n}$$

so that

$$\begin{aligned}
A &= \sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n} \\
&= \sum_{i=1}^{n} \left(1+\frac{2i}{n}\right)^2 \frac{2}{n} \\
&= \sum_{i=1}^{n} \left(1 + 2\cdot\frac{2i}{n} + \left[\frac{2i}{n}\right]^2\right)\frac{2}{n} \\
&= \sum_{i=1}^{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)\frac{2}{n} \\
&= \sum_{i=1}^{n} \frac{2}{n} + \frac{8i}{n^2} + \frac{8i^2}{n^3} \\
&= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{2}{n}\cdot n + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= 2 + \frac{4(n+1)}{n} + \frac{4(n+1)(2n+1)}{3n^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\int_1^3 x^2\,dx &= \lim_{n\to\infty}\left[2 + \frac{4(n+1)}{n} + \frac{4(n+1)(2n+1)}{3n^2}\right] \\
&= \left[2 + 4 + \frac{8}{3}\right] = \boxed{\frac{26}{3}}
\end{aligned}$$

We might as well check:

$$\int_1^3 x^2\,dx = \frac{1}{3}x^2\Big|_1^3 = \frac{1}{3}\left[3^3 - 1^3\right] = \frac{26}{3} \checkmark$$