THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

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MIDTERM EXAM #1
SOLUTIONS

22 Oct 2015   11:30–12:45

PROBLEM	GRADE	OUT OF
1		12
2		3
3		3
4		3
5		3
6		6
7		6
8		6
TOTAL:		42

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

/12

Problem 1: Evaluate the following:

/3 (a) $\displaystyle\int \left(x - \frac{2}{x} + e^{5x} + \pi \right) dx$

Integrate term by term:

$$\int \left(x - \frac{2}{x} + e^{5x} + \pi \right) dx = \boxed{\frac{1}{2}x^2 - 2\ln|x| + \frac{1}{5}e^{5x} + \pi x + C}$$

/3 (b) $\displaystyle\int_4^9 \frac{x - \sqrt{x}}{x^3}\, dx$

Simplify before integrating:

$$\int_4^9 \frac{x - x^{1/2}}{x^3}\, dx = \int_4^9 x^{-2} - x^{-5/2}\, dx = \left[-x^{-1} + \frac{2}{3}x^{-3/2} \right]_4^9$$

$$= \left(-\frac{1}{9} + \frac{2}{3} \cdot \frac{1}{27} \right) - \left(-\frac{1}{4} + \frac{2}{3} \cdot \frac{1}{8} \right) = \boxed{\frac{13}{162}}$$

/3 (c) $\displaystyle\int_0^\pi \frac{\sin x}{2 + \cos x}\, dx$

Substitution: $u = 2 + \cos x \implies du = -\sin x\, dx$

$$\implies \int_0^\pi \frac{\sin x}{2 + \cos x}\, dx = \int_3^1 \frac{-du}{u}\, dx = \int_1^3 \frac{du}{u}\, dx = \ln|u|\Big|_1^3 = \boxed{\ln 3}$$

/3 (d) $\displaystyle\int \frac{\ln x}{x^{10}}\, dx$

Integrate by parts:

$$u = \ln x \qquad dv = x^{-10}$$
$$du = \frac{1}{x}\, dx \qquad v = -\frac{1}{9}x^{-9}$$

$$\implies \int \frac{\ln x}{x^{10}}\, dx = uv - \int v\, du$$
$$= -\frac{1}{9}x^{-9} \ln x - \int -\frac{1}{9}x^{-9} \cdot \frac{1}{x}\, dx$$
$$= -\frac{\ln x}{9x^9} + \frac{1}{9} \int x^{-10}\, dx$$
$$= \boxed{-\frac{\ln x}{9x^9} - \frac{1}{81x^9} + C}$$

/3 **Problem 2:** Find $f'(x)$ where $f(x) = \displaystyle\int_{x^2}^{10} \frac{dz}{z^2+1}$.

First change the order of the limits

$$f(x) = -\int_{10}^{x^2} \frac{dz}{z^2+1}$$

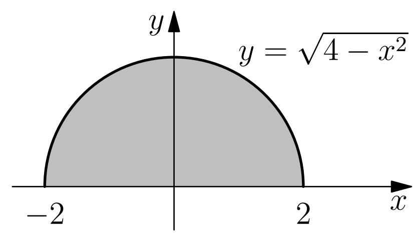
then use the Fundamental Theorem of Calculus (part 1) together with the chain rule:

$$f'(x) = -\frac{1}{(x^2)^2+1} \cdot 2x = \boxed{\frac{-2x}{x^4+1}}$$

/3 **Problem 3:** Evaluate $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$ by interpreting the integral as an area.

The integral represents the area of a half-circle of radius 2:

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx A = \frac{\pi \cdot 2^2}{2} = \boxed{2\pi}$$



/3 **Problem 4:** Write (but do not evaluate) a definite integral that represents the length of the graph of $y = x^3 + 2$ on the interval $[-2, 5]$.

We have $y' = 3x^2$. Using the arc length formula gives

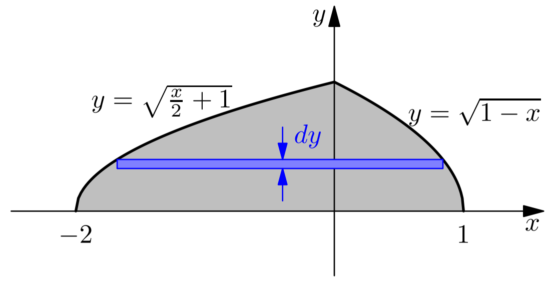
$$L = \int_{-2}^{5} \sqrt{1+(y')^2}\,dx = \boxed{\int_{-2}^{5} \sqrt{1+9x^4}\,dx}$$

/3 **Problem 5:** Find the average value of $y = \dfrac{1}{1+x^2}$ on the interval $[0, \frac{\pi}{4}]$.

Using the formula for the average value of a function on the interval $[0, \pi/4]$:

$$\begin{aligned}
\bar{y} &= \frac{1}{\pi/4} \int_0^{\pi/4} \frac{1}{1+x^2}\,dx \\
&= \frac{4}{\pi} \frac{1}{1+x^2}\,dx \\
&= \frac{4}{\pi} \arctan x \Big|_0^{\pi/4} \\
&= \frac{4}{\pi} \left(\arctan(\tfrac{\pi}{4}) - \arctan 0\right) = \boxed{\frac{4}{\pi} \arctan \frac{\pi}{4}}
\end{aligned}$$

/6 **Problem 6:** Find the area of the shaded region shown.



Solution 1:

This is easiest if we slice the area horizontally. We'll we need to solve each equation for x:

$$y = \sqrt{\frac{x}{2}+1} \implies x = 2y^2 - 2; \qquad y = \sqrt{1-x} \implies x = 1 - y^2.$$

The rectangles have area

$$dA = \left[(1-y^2) - (2y^2-2)\right] dy = (3-3y^2)\, dy$$

so

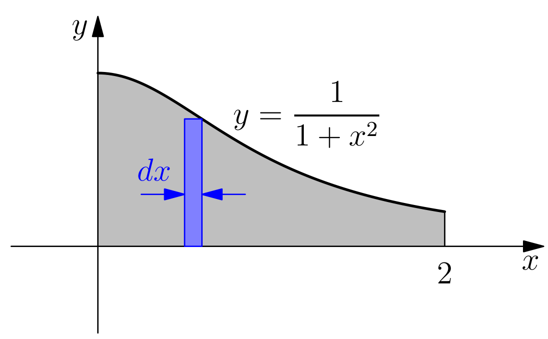
$$A = \int dA = \int_0^1 (3-3y^2)\, dy = 3y - y^3\Big|_0^1 = \boxed{2}$$

Solution 2:

Summing the areas under the graphs gives

$$A = \int_{-2}^{1} \sqrt{\frac{x}{2}+1}\, dx + \int_0^1 \sqrt{1-x}\, dx = \left[\frac{4}{3}\left(\frac{x}{2}+1\right)^{3/2}\right]_{-2}^{1} + \left[\frac{2}{3}(1-x)^{3/2}\right]_0^1 = \boxed{2}$$

/6 **Problem 7:** A solid object is made by revolving the shaded region below about the the y-axis. Find the volume of this object.



By "cylindrical shells":

$$dV = (2\pi x)(y)\, dx = 2\pi \frac{x}{1+x^2}\, dx$$

$$\begin{aligned} \implies V = \int dV &= 2\pi \int_0^2 \frac{x}{1+x^2}\, dx \qquad \text{substitute: } u = 1+x^2;\ du = 2x\, dx \\ &= \pi \int_1^5 \frac{du}{u} \\ &= \pi \ln|u|\Big|_1^5 = \boxed{\pi \ln 5} \end{aligned}$$

/6 **Problem 8:** Evaluate $\int_0^2 (4x+1)\,dx$ using the definition of the definite integral (i.e. by a limit of Riemann sums).

The following formulas may be useful: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

First construct the Riemann sum (area using n rectangles based on right endpoints):

$$\Delta x = \frac{2}{n} \qquad x_i = i\,\Delta x = \frac{2i}{n}$$

$$\implies S_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n}\left(4\cdot\frac{2i}{n}+1\right)\frac{2}{n}$$

$$= \frac{16}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum i = 1^n 1$$

$$= \frac{16}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n}\cdot n$$

$$= 8\cdot\frac{(n+1)}{n} + 2$$

Now let $n\to\infty$:

$$\int_0^2 (4x+1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty}\left[8\cdot\frac{(n+1)}{n}+2\right] = 8+2 = \boxed{10}$$