THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

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MIDTERM EXAM #1
SOLUTIONS

2 Oct 2014 11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		16
2		8
3		8
4		8
TOTAL:		40

/16 **Problem 1:** Evaluate the following:

/4 (a) $\displaystyle\int_0^1 (1-x)^9\,dx$

$$\begin{array}{l} u = 1-x \\ du = -dx \end{array} \implies \int_0^1 (1-x)^9\,dx = \int_1^0 -u^9\,du = \int_0^1 u^9\,du = \left.\frac{u^{10}}{10}\right|_{u=0}^1 = \boxed{\frac{1}{10}}$$

/4 (b) $\displaystyle\int x^{3/2}\ln x\,dx$

integrate by parts:

$$\begin{array}{ll} u = \ln x & dv = x^{3/2}\,dx \\ du = \frac{1}{x}\,dx & v = \frac{2}{5}x^{5/2} \end{array}$$

$$\begin{aligned} \implies \int x^{3/2}\ln x\,dx &= \frac{2}{5}x^{5/2}\ln x - \int \frac{2}{5}x^{5/2}\cdot\frac{1}{x}\,dx \\ &= \frac{2}{5}x^{5/2}\ln x - \int \frac{2}{5}x^{3/2}\,dx \\ &= \boxed{\frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2} + C} \end{aligned}$$

/4 (c) $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$$u = \sqrt{x} = x^{1/2}$$

$$du = \tfrac{1}{2}x^{-1/2}\,dx = \frac{dx}{2\sqrt{x}}$$

$$\implies \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^2 2e^u\,du = \left.2e^u\right|_1^2 = \boxed{2(e^2-e)}$$

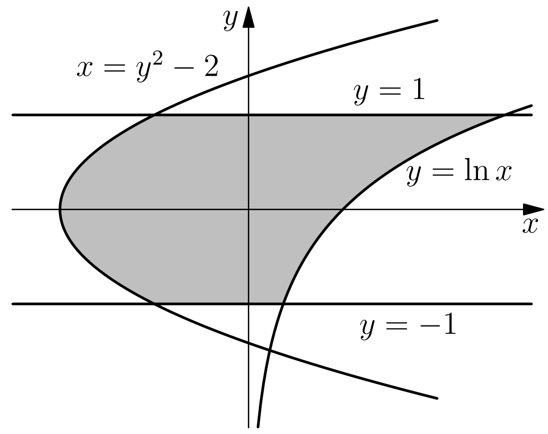
/4 (d) $\displaystyle\int \frac{\sin^2 x\,\cos x}{1+\sin^3 x}\,dx$

$$\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array} \implies \int \frac{\sin^2 x\,\cos x}{1+\sin^3 x}\,dx = \int \frac{u^2}{1+u^3}\,du$$

$$\begin{array}{l} w = 1+u^3 \\ dw = 3u^2\,du \end{array} \implies \int \frac{u^2}{1+u^3}\,du = \int \frac{1}{3}\frac{dw}{w} = \frac{1}{3}\ln|w| + C = \boxed{\frac{1}{3}\ln|1+\sin^3 x| + C}$$

/8

Problem 2: Find the area of the region bounded between the curves $y = \ln x$, $x = y^2 - 2$, $y = -1$ and $y = 1$.



By horizontal strips:

$$dA = \left(e^y - (y^2 - 2)\right)\, dy$$

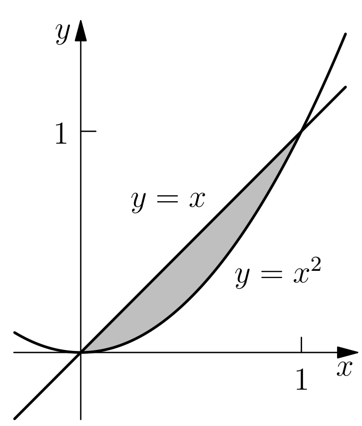
$$\implies A = \int_{-1}^{1} (e^y - y^2 + 2)\, dy = \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1}$$

$$= \left(e - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right)$$

$$= \boxed{e - \frac{1}{e} + \frac{10}{3}}$$

/8

Problem 3: The region enclosed by the curves $y = x$ and $y = x^2$ is revolved about the the x-axis. Find the volume of the resulting solid.



Revolving vertical strips generates "washers" perpendicular to the x-axis:

$$dV = \pi(R^2 - r^2)\, dx = \pi\left(x^2 - (x^2)^2\right)\, dx$$

$$\implies V = \int dV = \int_0^1 \pi\left(x^2 - x^4\right)\, dx = \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \boxed{\frac{2\pi}{15}}$$

Alternatively, revolving horizontal strips generates "cylindrical shells" parallel to the x-axis:

$$dV = 2\pi y\,(\sqrt{y} - y)\, dy$$

$$\implies V = \int dV = \int_0^1 2\pi\left(y^{3/2} - y^2\right)\, dy = 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \boxed{\frac{2\pi}{15}}$$

/8 **Problem 4:** Evaluate $\int_0^3 (x^2 - 2)\,dx$ using the definition of the definite integral (i.e. by Riemann sums).

The following formulas might be useful: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

First construct the Riemann sum (area using n rectangles based on right endpoints):

$$\Delta x = \frac{3}{n} \qquad x_i = i\,\Delta x = \frac{3i}{n}$$

$$\begin{aligned}
\implies S_n = \sum_{i=1}^{n} f(x_i)\,\Delta x &= \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 2\right]\frac{3}{n} \\
&= \sum_{i=1}^{n} \frac{27i^2}{n^3} - \frac{6}{n} \\
&= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{6}{n}\sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\cdot n \\
&= \frac{9}{2}\cdot\frac{2n^3+3n^2+n}{n^3} - 6
\end{aligned}$$

Now let $n \to \infty$:

$$\begin{aligned}
\int_0^3 (x^2-2)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= \lim_{n\to\infty}\frac{9}{2}\cdot\frac{2n^3+3n^2+n}{n^3} - 6 = 9 - 6 = \boxed{3}
\end{aligned}$$