MATH 1230: Quiz #7 – SOLUTIONS

/5 **Problem 1:** Find the quadratic polynomial approximation of $f(x) = \sqrt[3]{x}$ centered at $a = 8$, and use it to approximate the value of $\sqrt[3]{7.5}$.

We need the values of the derivatives of $f(x)$ at $x = 8$:

$$\begin{aligned} f(x) &= x^{1/3} && f(8) = 8^{1/3} = 2 \\ f'(x) &= \tfrac{1}{3}x^{-2/3} \quad \Longrightarrow && f'(8) = \tfrac{1}{3}8^{-2/3} = \tfrac{1}{3}2^{-2} = \tfrac{1}{12} \\ f''(x) &= -\tfrac{2}{9}x^{-5/3} && f''(8) = -\tfrac{2}{9}8^{-5/3} = -\tfrac{2}{9}2^{-5} = -\tfrac{1}{144} \end{aligned}$$

The quadratic approximation is then

$$\begin{aligned} P_2(x) &= f(8) + f'(8)(x-8) + \tfrac{1}{2}f''(8)(x-8)^2 \\ &= \boxed{2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2.} \end{aligned}$$

This gives

$$\sqrt[3]{7.5} = f(7.5) \approx P(7.5) = 2 + \tfrac{1}{12}(-0.5) - \tfrac{1}{288}(-0.5)^2 \approx \boxed{1.957465}$$

(Note that this agrees to 4 decimal places with the exact value.)

/5 **Problem 2:** Find Taylor series for the function $f(x) = \dfrac{1}{x^2}$ centered at $a = -1$.

We need the values of the derivatives of $f(x)$ at $x = -1$:

$$\begin{aligned} f(x) &= x^{-2} && f(-1) = 1 \\ f'(x) &= -2x^{-3} && f'(-1) = 2 = 2! \\ f''(x) &= 2 \cdot 3x^{-4} \quad \Longrightarrow && f''(-1) = 2 \cdot 3 = 3! \\ f'''(x) &= -2 \cdot 3 \cdot 4x^{-5} && f'''(-1) = 2 \cdot 3 \cdot 4 = 4! \\ f^{(4)}(x) &= 2 \cdot 3 \cdot 4 \cdot 5x^{-5} && f^{(4)}(-1) = 2 \cdot 3 \cdot 4 \cdot 5 = 5! \end{aligned}$$

The pattern is fairly clear: $f^{(n)}(-1) = (n+1)!$. The Taylor series about $a = -1$ is then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(x+1)^n \\ &= \boxed{\sum_{n=0}^{\infty}(n+1)(x+1)^n = 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + 5(x+1)^4 + \cdots} \end{aligned}$$