MATH 1230: Quiz #4 – SOLUTIONS

/3 **Problem 1:** Evaluate: $\displaystyle\int \frac{x+5}{x^3-2x^2-3x}\,dx$

partial fractions:

$$\frac{x+5}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1} = \frac{A(x-3)(x+1)+Bx(x+1)+Cx(x-3)}{x(x-3)(x+1)}$$

$$\begin{array}{lll} x=0: & 5=-3A & \implies A=-\frac{5}{3} \\ x=3: & 8=12B & \implies B=\frac{2}{3} \\ x=-1: & 4=4C & \implies C=1 \end{array}$$

$$\int\left(\frac{-5/3}{x}+\frac{2/3}{x-3}+\frac{1}{x+1}\right)dx = \boxed{-\tfrac{5}{3}\ln|x|+\tfrac{2}{3}\ln|x-3|+\ln|x+1|+C}$$

/3 **Problem 2:** Evaluate the following integral (or show that it diverges): $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$

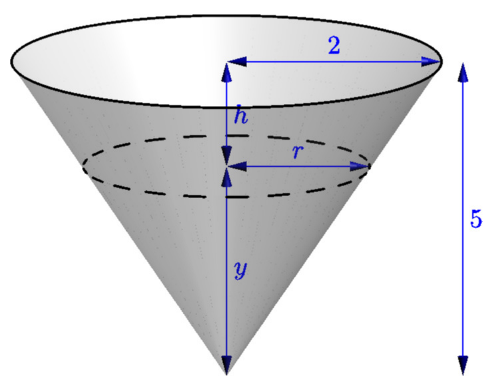
Substitution: $u=\ln x,\ du=\frac{1}{x}\,dx$:

$$\int\frac{1}{x\ln x}\,dx = \int\frac{du}{u} = \ln|u| = \ln|\ln x|$$

$$\begin{aligned}\implies \int_1^\infty \frac{1}{x\ln x}\,dx &= \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx \\ &= \lim_{b\to\infty}\Big[\ln|\ln x|\Big]_2^b \\ &= \lim_{b\to\infty}\left[\ln(\ln b)-\ln(\ln 2)\right] = \infty \implies \boxed{\text{the integral diverges}}\end{aligned}$$

/4 **Problem 3:** A water tank is shaped like an inverted cone (i.e. the vertex of the cone is at the bottom), with height 5 m and radius 2 m. If the tank is full of water how much work is required to pump all the water to the level of the top of the tank? Express your answer in terms of the density of water, ρ, and the acceleration of gravity, g.

When the water depth is y, decreasing the depth by dy involves lifting a thin cylinder of volume $V=\pi r^2\,dy$ by a height $h=5-y$. The incremental work required is thus



$$dW = mgh = \rho V g(5-y) = \rho\pi r^2\,dy\,g(5-y).$$

By similar triangles, we have

$$\frac{r}{y}=\frac{2}{5} \implies r=\tfrac{2}{5}y.$$

$$\implies dW = \pi\rho g\left(\tfrac{2}{5}y\right)^2(5-y)\,dy.$$

Thus the total work to pump all the water to the top is

$$\begin{aligned} W = \int dW &= \int_0^5 \pi\rho g\left(\tfrac{2}{5}y\right)^2(5-y)\,dy \\ &= \tfrac{4}{25}\pi\rho g\int_0^5 y^2(5-y)\,dy \\ &= \tfrac{4}{25}\pi\rho g\underbrace{\int_0^5 5y^2-y^3\,dy}_{\frac{5}{3}y^3-\frac{1}{4}y^4\Big|_0^5=\frac{625}{12}} = \boxed{\tfrac{25}{3}\pi\rho g}\end{aligned}$$