THOMPSON RIVERS UNIVERSITY

MATH 1230
Calculus 2 for Engineering

Instructor: Richard Taylor

MIDTERM EXAM #1
SOLUTIONS

13 Feb. 2019 10:00–11:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		14
2		4
3		5
4		10
5		6
TOTAL:		39

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Problem 1: Evaluate the following:

/3 (a) $\displaystyle\int \left(\sqrt{x} - \frac{1}{5x} + \pi^2 + e^{-3x}\right) dx$

$$\int \left(x^{1/2} - \frac{1}{5}\frac{1}{x} + \pi^2 + e^{-3x}\right) dx = \boxed{\tfrac{2}{3}x^{3/2} - \tfrac{1}{5}\ln|x| + \pi^2 x - \tfrac{1}{3}e^{-3x} + C}$$

/3 (b) $\displaystyle\int x^{1/3}(2-x)^2\, dx$

$$\begin{aligned}\int x^{1/3}(4 - 4x + x^2)\, dx &= \int 4x^{1/3} - 4x^{4/3} + x^{7/3}\, dx \\ &= \boxed{3x^{4/3} - \tfrac{12}{7}x^{7/3} + \tfrac{3}{10}x^{10/3} + C}\end{aligned}$$

/4 (c) $\displaystyle\int_0^{\pi/2} x \sin x\, dx$

Integrate by parts:

$$\begin{matrix} u = x & dv = \sin x\, dx \\ du = dx & v = -\cos x \end{matrix} \implies \int_0^{\pi/2} x \sin x\, dx = \underbrace{-x\cos x\Big|_0^{\pi/2}}_{0} + \underbrace{\int_0^{\pi/2} \cos x\, dx}_{\sin x\big|_0^{\pi/2} = 1} = \boxed{1}$$

/4 (d) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}e^{\sqrt{x}}}$

Subsitution:

$$u = \sqrt{x} = x^{1/2}, \quad du = \tfrac{1}{2}x^{-1/2}$$

$$\int_1^4 \frac{dx}{\sqrt{x}e^{\sqrt{x}}} = \int_1^2 \frac{2\, du}{e^u} = \int_1^2 2e^{-u}\, du = \Big[-2e^{-u}\Big]_1^2 = \boxed{2(e^{-1} - e^{-2})}$$

/4 **Problem 2:** Let $f(x) = \int_0^x \sqrt{4-t^2}\,dt$. Evaluate the following:

(a) $f(2)$

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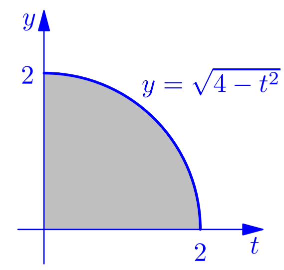
$y = \sqrt{4-t^2} \implies t^2 + y^2 = 2^2$ is a circle of radius 2

$\implies$ the area under the graph is $\frac{1}{4}\pi(2^2) = \boxed{\pi}$



(b) $f'(2)$

/2

By the fundamental theorem of calculus:

$$f'(x) = \sqrt{4-x^2} \implies f'(2) = \boxed{0}$$

/5 **Problem 3:** A reservoir supplies water to an industrial park. At 10:00AM on a certain day the reservoir contains 8000 gallons of water. The rate at which the reservoir supplies water is given by the formula $r(t) = 10 + \sqrt{t}$ [gal/min] where t is measured in minutes since 10:00AM. How much water is left in the reservoir at 12 noon?

Let $V(t)$ be the volume of water in the reservoir. Then $V'(t) = -r(t)$ and so

$$V(120) - V(0) = \int_0^{120} V'(t)\,dt = -\int_0^{120} (10 + t^{1/2})\,dt = -\left[10t + \tfrac{2}{3}t^{3/2}\right]_0^{120} \approx -2076 \text{ gallons}$$

$$\implies V(120) = V(0) - 2076 = 8000 - 2076 \approx \boxed{5924 \text{ gallons}}$$

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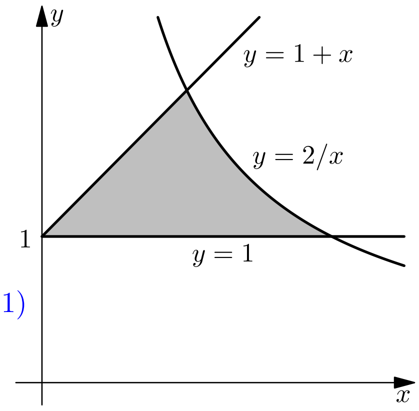
Problem 4: Consider the shaded region shown in the graph below.



(a) Calculate the area of the shaded region.

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The upper intersection point is at

$$\begin{aligned} 1 + x &= \frac{2}{x} \\ \implies x^2 + x - 2 &= 0 = (x+2)(x-1) \\ \implies x = 1 &\implies y = 2. \end{aligned}$$

Re-arrange the equations as follows:

$$y = 1 + x \implies x = y - 1 \quad \text{and} \quad y = \frac{2}{x} \implies x = \frac{2}{y}.$$

Then by "horizontal slices":

$$dA = \left(\frac{2}{y} - (y-1)\right)\, dy$$

$$\begin{aligned} A = \int dA &= \int_1^2 \left(\frac{2}{y} - y + 1\right)\, dy \\ &= 2\ln|y| - \tfrac{1}{2}y^2 + y\Big|_1^2 \\ &= (2\ln 2 - 2 + 2) - (0 - \tfrac{1}{2} + 1) = \boxed{2\ln 2 - \tfrac{1}{2} \approx 0.886} \end{aligned}$$

(b) A solid object is formed by revolving the shaded region about the y-axis. Calculate the volume of this object.

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Horizontal slices generate "washers" with outer radius $R = \frac{2}{y}$, inner radius $r = 1 - y$, and volume

$$dV = \pi(R^2 - r^2)\, dy = \pi\left(\frac{4}{y^2} - (1-y)^2\right)\, dy$$

$$\begin{aligned} \implies V = \int dV &= \pi \int_1^2 \left(\frac{4}{y^2} - (1-y)^2\right)\, dy \\ &= \pi \int_1^2 \left(4y^{-2} - 1 + 2y - y^2\right)\, dy \\ &= \pi\Big[-4y^{-1} - y + y^2 - \tfrac{1}{3}y^3\Big]_1^2 \\ &= \pi\left(\big[-2 - 2 + 4 - \tfrac{8}{3}\big] - \big[-4 - 1 + 1 - \tfrac{1}{3}\big]\right) = \boxed{\frac{5\pi}{3}} \end{aligned}$$

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Problem 5: Use the definition of the definite integral (as a limit of Riemann sums) to evaluate $\int_0^2 (2x - 3x^2)\,dx$. The following formulas might be useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Let $f(x) = 2x - 3x^2$. With $\Delta x = \dfrac{2}{n}$ and $x_i = i\Delta x = \dfrac{2i}{n}$, we have

$$\begin{aligned}
\int_0^2 f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\frac{2i}{n} - 3\left(\frac{2i}{n}\right)^2 \right] \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\frac{4i}{n} - \frac{12i^2}{n^2} \right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left[\frac{4}{n} \sum_{i=1}^{n} i - \frac{12}{n^2} \sum_{i=1}^{n} i^2 \right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left[\frac{4}{n} \frac{n(n+1)}{2} - \frac{12}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\
&= \lim_{n\to\infty} \left[\frac{4(n+1)}{n} - \frac{4(n+1)(2n+1)}{n^2} \right] \\
&= 4 - 8 = \boxed{-4}
\end{aligned}$$

Check:

$$\int_0^2 (2x - 3x^2)\,dx = x^2 - x^3 \Big|_0^2 = 4 - 8 = -4 \ \checkmark$$