MATH 1230 Lecture Notes

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1 Introduction

Reading mathematics is a fairly advanced skill. Most students find that an interactive lecture format is an easier way to get acquainted with new mathematical ideas. But reading math is a skill worth learning, and this seems as good a time as any to start working on it.

The dates indicated here are the days I would have covered the material in class, and I've put them here to help you pace your work. In the right-hand margin I have indicated the relevant sections of the textbook. In some places I have suggested specific sections you should definitely read, and exercises you should attempt as you progress. I will add to this document as we go, so **check back here often**.

I've purposefully made these notes brief and to the point. You should use the textbook for supplemental information, and especially be sure to do the assigned readings. I aim to get you solving relevant problems on your own as quickly as possible. If you read these notes, do the assigned readings, and still find it difficult to get started on the problems, send me an email ASAP. I will help however I can, and your feedback will help me to make these notes better.

March 23

2 Infinite Series

Sec. 10.1

We have seen that many scientific quantities of interest can be written (and calculated) as definite integrals. A definite integral is really an infinite sum of a peculiar sort (limit of Riemann sums).

Another, closely related kind of infinite sum is called an *infinite series*. An example is the following:

$$\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots . \tag{1}$$

With sigma notation we can write this quantity in shorter form as

$$\sum_{n=1}^{\infty} \frac{1}{5^n}. \tag{2}$$

Despite there being infinitely many terms, the value of this sum is finite. This isn't really surprising, because the individual terms get small very quickly ($1/5^4 = 0.0016$). In fact you might guess that we can approximate the sum by adding only a *finite* number of terms:

$$\sum_{n=1}^{\infty} \frac{1}{5^n} \approx \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} = 0.2496. \tag{3}$$

We can get a more accurate approximation by including more terms:

$$\sum_{n=1}^{\infty} \frac{1}{5^n} \approx \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} = 0.24992. \tag{4}$$

This is a good idea, and we'll use it later on.

For now, we are concerned with interpreting eq. (1) as representing an *exact* value. As usual, this means we need to interpret the "∞" in the sense of a *limit* as the number of terms (say, N) goes to infinity:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{5^n} &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{5^n} && (5) \\ &= \lim_{N\to\infty} \left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^N} \right). && (6) \end{aligned}$$

Notice the similarity with Riemann sums, which we used at the beginning of the course to calculate areas.

Reading Assignment: Sec. 10.1, Examples 7 and 8.

Exercises: Approximate the value of the each infinite series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ (recall that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$, with the special case $0! = 1$)

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3 Geometric Series

Sec. 10.3

Unlike integrals, it usually isn't possibly to evaluate an infinite series exactly. In fact, exact evaluating of series isn't really the point, and usually we won't even try.

However, one important exception is the case of *geometric series*. A geometric series has the specific form

$$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \cdots \tag{7}$$

where r is some constant.

Exercise: Look at eq. (1) and confirm that this is a geometric series with $r = 1/5$.

Notice that if $|r| > 1$ then the terms in eq. (7) grow in magnitude, so the value of the sum obviously is not finite. In this case we say the series **diverges**. In fact the series diverges also when $|r| = 1$ (why?)

However, if $|r| < 1$ then terms in eq. (7) rapidly get smaller, and in fact the series **converges** to a finite value that we can calculate as follows:

$$\boxed{\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \qquad\qquad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.} \tag{8}$$

We can use these formulas to find the exact value of the series in eq. (1):

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = \frac{\frac{1}{5}}{1 - \frac{1}{5}} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}. \tag{9}$$

Reading Assignment: pp. 663–664.

Exercises: Sec. 10.3, #21, 23, 25, 27, 29, 41

March 25

4 Convergence Tests

With the exception of geometric series (which are a very special case), we generally aren't able to calculate the exact value of an infinite series—and in fact, unlike our work on definite integrals, *evaluating* series isn't really the point.

Consider the following infinite series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \cdots . \tag{10}$$

If this expression is to represent the answer to some quantitative scientific question, then we need to know whether the series **converges**, i.e.

$$\text{does } \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} \text{ exist?}$$

If it does not, then the we say the series **diverges**; for practical purposes this means the series is pretty useless, since it doesn't represent a finite quantity. (Divergent series aren't entirely useless, though; they play an important role in lots of areas of math, many of which are relevant to problems in engineering.)

If an infinite series converges, that means it represents some actually quantity. If we want to evaluate that quantity, we can just add as many terms in the series as necessary to get a good approximation (3 decimal places is plenty for most engineering applications).

Intuitively, you might think that the series in eq. (10) converges because the individual terms get smaller (**try calculating** $1/10!$, which is very small indeed!). That intuition is partly correct, but convergence is a more subtle issue than that. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{11}$$

is actually divergent (see Theorem 10.10 in the textbook) in that $\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n}$ does not exist. The terms in this series get smaller, just not *quickly* enough to get convergence. We saw something similar in convergence of improper integrals.

To work with series we need ways to distinguish between convergent (useful) series and divergent (mostly useless) series. Chapter 10 of our textbook covers many such tests; we will only look at a couple of these.

4.1 Ratio Test

Sec. 10.7

Often the simplest test for convergence of a series is the ratio test.

Theorem. *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series, and let* $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. *Then:*

1. *If* $r < 1$ *then the series converges.*
2. *If* $r > 1$ *then the series diverges.*
3. *If* $r = 1$ *then the test is inconclusive.*

This theorem works by looking at the ratio between successive terms a_n and a_{n+1} in the limit as $n \to \infty$. If this ratio is less than 1, this means the series *eventually* looks like a convergent geometric series of the form $\sum_{n=1}^{\infty} ar^n$, and we conclude that our series converges (although we don't get to evaluate its sum explicitly, as we could with a true geometric series). If $r > 1$, this means the terms in the sum are eventually growing larger (**why?**) so the series can't possibly converge.

Note that it really doesn't matter whether the series starts at $n = 0$ or $n = 1$ or anything else. For testing convergence, what matters is the eventual *limiting behavior* of the individual terms. For this, the first few terms (and even the first few thousands of terms!) are irrelevant.

Example: Does the infinite series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converge?

Solution: Identify $a_n = \frac{1}{n!}$ and apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n\cdot(n+1)} = \frac{1}{n+1} \tag{12}$$

and so

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0. \tag{13}$$

Since $r < 1$ we conclude that the series converges.

In practice, this means it is possible to accurately *approximate* the value of the series by summing sufficiently many terms, e.g.

$$\sum_{n=0}^{\infty}\frac{1}{n!} \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.717.$$

Because the terms get small so quickly, the error in this approximation is roughly the size of the first term we left out, i.e. $\frac{1}{6!} \approx 0.001$. If this isn't good enough, just include more terms until the error estimate is as small as you would like.

Reading Assignment: Sec. 10.7, Example 1.

Exercises: Sec. 10.7, #9, 13, 15, 17, 21

March 30

5 Power Series

Sec. 11.2

Now that we can represent numerical quantities using infinite series, it becomes possible to represent a *function* $f(x)$ by a series that depends on x. At first this might seem strange and even unnecessarily complicated, but actually this idea is extremely useful and important throughout higher mathematics, including much of the math that touches on engineering.

For example, the infinite series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots \tag{14}$$

defines a function $f(x)$. For some values of x this series will converge; these x constitute the domain of the function. For other values of x it may be that the series does not converge; these are outside the domain of f.

In general, a **power series** is a function of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n. \tag{15}$$

Here a is a constant (called the **center** of the power series), often equal to 0; the numbers c_n are called the **coefficients** of the series. Power series aren't the *only* way to represent a function as a series (e.g. later you might encounter Fourier series) but they are the most common and useful.

The domain of a power series is always in interval of the form $(a-R, a+R)$ called the **interval of convergence**. The number R (which might be ∞) is called the **radius of convergence**. To find the interval of convergence for a given power series, the ratio test will often work.

Example: Find the radius and interval of convergence of the power series $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$.

Solution: The center of this power series is $a = 0$. Its coefficients are $c_n = (-1)^n/5^n$. To use the ratio test, calculate the limiting ratio of the $(n+1)$th to the nth term:

$$r = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}/5^{n+1}}{(-1)^n x^n/5^n} \right| \tag{16}$$

$$= \lim_{n\to\infty} \left| \frac{x}{5} \right| \tag{17}$$

$$= |x|/5. \tag{18}$$

A convergent series requires $r < 1$, i.e.

$$|x|/5 < 1 \iff |x| < 5 \iff -5 < x < 5. \tag{19}$$

That is, the interval of convergence is $(-5, 5)$. Since the center is $a = 0$, the radius of convergence is $R = 5$.

The ratio test also guarantees that this power series *diverges* if $|x| > 5$ (**why?**). Note that the ratio test is inconclusive at $x = \pm 5$ so we don't actually know if the endpoints of the interval are included in the domain of f. Sometimes a different convergence test can be used to settle this; we won't worry about it.

Exercises: Sec. 11.2; #9, 11, 15, 19, 23, 25, 29

6 Power Series via Geometric Series

Sec. 11.2

The power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{20}$$

is actually a geometric series (with ratio $r = x$). So we know that this series diverges if $|x| \geq 1$, converges if $|x| < 1$, and in this case its value is the function

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad (|x| < 1). \tag{21}$$

Similarly,

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x} \qquad (|x| < 1). \tag{22}$$

We can use this to find series representations of some related functions...

Example: Find a power series and its interval and radius of convergence for the function

$$f(x) = \frac{1}{1+4x^2}.$$

Solution: We need to get the function in the form $1/(1-r)$ so we can recognize it as a geometric series with ratio r. We have

$$f(x) = \frac{1}{1+4x^2} = \frac{1}{1-(-4x^2)}. \tag{23}$$

This is in the form $1/(1-r)$ with $r = -4x^2$, so

$$f(x) = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} (-4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} = 1 - 4x^2 + 16x^4 - 64x^6 + \cdots \tag{24}$$

Because this a geometric series, it converges if

$$|r| < 1 \iff 4x^2 < 1 \iff x^2 < \frac{1}{4} \iff |x| < \frac{1}{2}. \tag{25}$$

Thus the interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$. This power series is centered at $a = 0$ and its radius of convergence is $R = \frac{1}{2}$.

Reading Assignment: Sec. 11.2, Example 3

Exercises: Sec. 11.2; #41, 43, 45

April 1

7 Taylor Polynomials

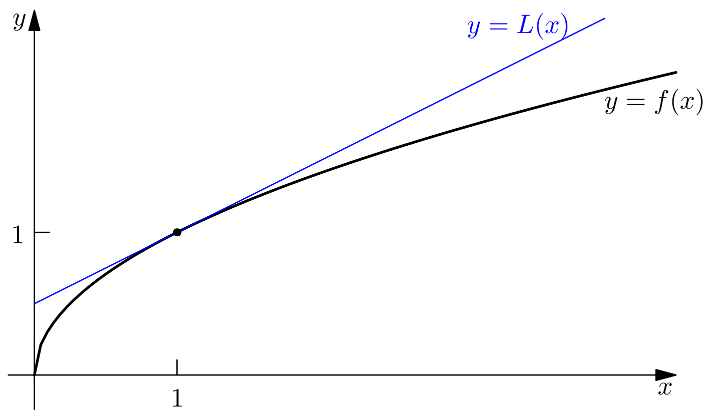
Sec. 11.1

Recall that for a given function $f(x)$, its **linear approximation** based at $x = a$ is the function

$$L(x) = f(a) + f'(a)(x - a). \tag{26}$$

The graph of $L(x)$ is the graph of the line tangent to the graph of $f(x)$ at $x = a$. For values of x close to a, values of $L(x)$ are quite close to values of $f(x)$, so $L(x)$ can be useful for making approximations.

For example, the figure below shows the graph of $f(x) = \sqrt{x}$ together with its linear approximation $L(x)$ based at $x = 1$. Notice how close to two graphs are when x is close to 1.



Note that $L(x)$ is, by definition, the linear function that has both the same value and same derivative as f at $x = a$. We can extend this idea to make a *quadratic* approximation of f, by finding the quadratic function that has the same value and same first and *second* derivatives as f at $x = a$. This gives the function

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2. \tag{27}$$

(**you should check for yourself that** $P_2(a) = f(a)$**,** $P_2'(a) = f'(a)$ **and** $P_2''(a) = f''(a)$**.**)

Example: Find the quadratic approximation of $f(x) = \sqrt{x}$ based at $x = 1$.

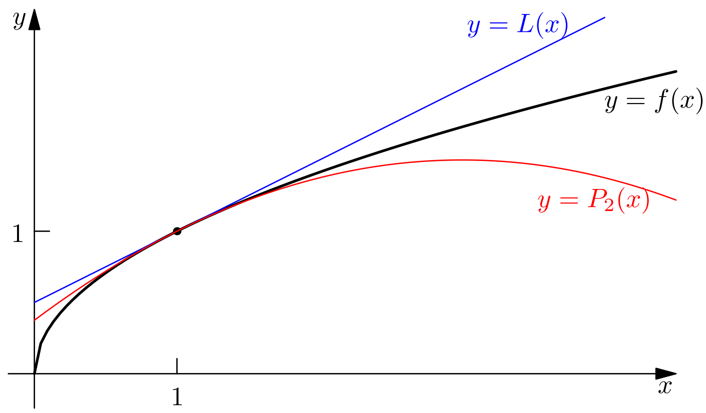
Solution: We have

$$\begin{aligned} f(x) &= \sqrt{x} = x^{1/2} & & & f(1) &= 1 \\ f'(x) &= \tfrac{1}{2}x^{-1/2} & &\Longrightarrow & f'(1) &= \tfrac{1}{2} \\ f''(x) &= -\tfrac{1}{4}x^{-3/2} & & & f''(1) &= -\tfrac{1}{4} \end{aligned} \tag{28}$$

So with $a = 1$ eq. (27) gives the quadratic approximation

$$P_2(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2 \tag{29}$$
$$= \boxed{1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2.} \tag{30}$$

The figure below shows the graph of f together with quadratic approximation P_2 and the linear approximation $L = P_1$. Notice how much closer the graph of P_2 is to the graph of f, at least where x is close to 1.



Reading Assignment: pp. 709–710

Extending this idea further, we can make an nth-order *polynomial approximation* of $f(x)$, based at $x = a$:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{31}$$

We can write this more compactly using sigma notation:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{32}$$

The polynomial P_n is sometimes called the nth-order **Taylor polynomial** for f. It is the unique degree-n polynomial whose derivatives (evaluated at $x = a$) all agree with those of f, up to order n.

Example: Find the 5th-order Taylor polynomial for $\sin x$ based at $x = 0$, and use it to approximate $\sin(0.3)$.

__**Solution:**__ We first need to find the values of the derivatives of $f(x) = \sin x$ up to order 5, all evaluated at $x = 0$:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ f^{(5)}(x) &= \cos x \end{aligned} \quad \Longrightarrow \quad \begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ f^{(5)}(0) &= 1 \end{aligned} \tag{33}$$

Putting these into eq. (32) with $a = 0$ gives

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}5^4 \tag{34}$$

$$= \boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5.} \tag{35}$$

Since the values of $P_5(x)$ are close to those of $f(x)$ when x is close to $a = 0$, we can approximate

$$\sin(0.3) \approx P_5(0.3) = 0.3 - \frac{1}{3!}(0.3)^3 + \frac{1}{5!}(0.3)^5 \approx \boxed{0.2955} \tag{36}$$

(**check this against the actual value of** $\sin(0.3)$ **obtained using a calculator!**)

Notice that evaluating $P_5(0.3)$ requires only *arithmetic*, i.e. multiplication, division, addition and subtraction. Because of this, Taylor polynomials play a key role in programming computers to evaluate mathematical functions: on most computer hardware the only built-in mathematical operations are the arithmetic ones.

__**Reading Assignment:**__ pp. 711–712

__**Exercises:**__ Ch. 11; #9, 11, 25, 33, 35, 37

April 2

8 Taylor Series

Sec. 11.3

If $f(x)$ is infinitely differentiable then the Taylor polynomial in eq. (32) becomes a better and better approximation of f as the order n increases. In the limit as $n \to \infty$ the Taylor polynomial becomes an infinite series (called the **Taylor series for f centered at a**):

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{37}$$

The particular case where $a = 0$ is the most common; in this case the series is called a **Maclaurin series**:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{38}$$

For most functions you are likely to encounter this series will agree *exactly* with $f(x)$, provided that the series converges.

Taylor and Maclaurin series are fundamental tools of applied math, including many areas of math that touch on engineering applications. It will probably seem unnatural at first, but representing a function by its Taylor series can be extremely useful

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Example: Find the Maclaurin series for $f(x) = e^x$.

Solution: To use eq. (38) we need to know the values of *all* the derivatives $f^{(k)}(0)$. This turns out not to be difficult here:

$$\begin{aligned}
f(x) &= e^x & & & f(0) &= 1 \\
f'(x) &= e^x & & & f'(0) &= 1 \\
f''(x) &= e^x & &\Longrightarrow & f''(0) &= 1 \\
f'''(x) &= e^x & & & f'''(0) &= 1 \\
&\vdots & & & &\vdots
\end{aligned} \tag{39}$$

There is an obvious pattern here: evidently $f^{(k)}(0) = 1$ for all $k = 0, 1, 2, \ldots$. Putting these into eq. (38) gives the Maclaurin series

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots \tag{40}$$

$$= 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots \tag{41}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^k}{k!}.} \tag{42}$$

It is easy to show (using the ratio test) that this series converges for *all* x. It can also be shown (see pp. 738–739 of the text) that the series actually converges to e^x. In other words, for any calculation involving e^x it is possible (and often quite useful) to replace e^x with the power series above.

Example: Find the Maclaurin series for $f(x) = \cos x$.

Solution: Again, we need to find the values of the derivatives $f^{(k)}(0)$ for $k = 0, 1, 2, \ldots$

$$\begin{aligned}
f(x) &= \cos x & & & f(0) &= 1 \\
f'(x) &= -\sin x & & & f'(0) &= 0 \\
f''(x) &= -\cos x & &\Longrightarrow & f''(0) &= -1 \\
f'''(x) &= \sin x & & & f'''(0) &= 0 \\
f^{(4)}(x) &= \cos x & & & f^{(4)} &= 1 \\
&\vdots & & & &\vdots
\end{aligned} \tag{43}$$

If you don't see the pattern already, just calculate a few more terms until it becomes obvious. Once the pattern is clear then in principle we know all the values of $f^{(k)}(0)$. Putting these into eq. (38) gives the Maclaurin series

$$\cos x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots \tag{44}$$

$$= 1 + 0 - \frac{1}{2!} x^2 + 0 - \frac{1}{4!} x^4 + \cdots \tag{45}$$

$$= \boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} \tag{46}$$

It isn't strictly necessary, but with a little cleverness it's possible to use sigma notation to write an explicit formula for this series. Noting that the series contains only even-order terms, it helps to write the order of each term as $2m$ (with $m = 0, 1, 2, \ldots$) and then the series can be written

$$\boxed{\sum_{m=0}^{\infty}(-1)^m \frac{x^{2m}}{(2m)!}.} \tag{47}$$

Notice how the factor $(-1)^m$ takes care of the alternating sign of each term.

Example: Find the Maclaurin series for $f(x) = \cos(x^3)$.

Solution: We could use eq. (38) directly, but there is an easier method here. Since we already know the Maclaurin series for $\cos x$,

$$\cos x = \sum_{m=0}^{\infty}(-1)^m \frac{x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{48}$$

we can just substitute x^3 in place of x to get

$$\cos(x^3) = \sum_{m=0}^{\infty}(-1)^m \frac{(x^3)^{2m}}{(2m)!} = \boxed{\sum_{m=0}^{\infty}(-1)^m \frac{x^{6m}}{(2m)!} = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots} \tag{49}$$

See the textbook p. 740 for a table of Taylor and Maclaurin series of some common functions. These can be used to help find series representations of related functions, following the example above.

Example: Find the Maclaurin series for $f(x) = \sin x$.

Solution: It wouldn't be hard to use eq. (38) directly, but here is alternative that is sometime useful. We know that $\sin x = -\frac{d}{dx}\cos x$. Since we already know the Maclaurin series for $\cos x$, we can just differentiate the series for $\cos x$ (it is generally valid to differentiate and integrate power series term by term):

$$\sin x = -\frac{d}{dx}\cos x \tag{50}$$

$$= -\frac{d}{dx}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] \tag{51}$$

$$= -\left[0 - \frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \cdots\right] \tag{52}$$

$$= \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{m=0}^{\infty}(-1)^m \frac{x^{2m+1}}{(2m+1)!}} \tag{53}$$

Exercises: Sec. 11.3; #9, 11, 25, 35, 36, 37, 77

April 3

9 Applications of Taylor Series

Sec. 11.4

Example: Find an infinite series whose value is $\sqrt{e}$, and use this series to calculate this quantity to 3 decimal places.

Solution: We have the following Maclaurin series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{54}$$

So then

$$\sqrt{e} = e^{1/2} = \sum_{k=0}^{\infty} \frac{(1/2)^k}{k!} = \boxed{\sum_{k=0}^{\infty} \frac{1}{2^k k!} = 1 + \frac{1}{2} + \frac{1}{2^2 2!} + \frac{1}{2^3 3!} + \cdots} \tag{55}$$

Evaluating the individual terms gives

$$\sqrt{e} = 1 + \frac{1}{2} + \underbrace{\frac{1}{2^2 2!}}_{0.125} + \underbrace{\frac{1}{2^3 3!}}_{0.0208} + \underbrace{\frac{1}{2^4 4!}}_{0.0026} + + \underbrace{\frac{1}{2^5 5!}}_{0.00026} \cdots \tag{56}$$

The terms rapidly get smaller, so any *additional* terms certainly won't affect the value beyond the 3rd decimal place (this is a rough way of figuring, but good enough for now; a more rigorous justification relies on the Taylor polynomial remainder, see p. 715). So we can conclude that

$$\sqrt{e} \approx 1 + \frac{1}{2} + \frac{1}{2^2 2!} + \frac{1}{2^3 3!} + \frac{1}{2^4 4!} + \frac{1}{2^5 5!} \approx \boxed{1.649} \tag{57}$$

You probably wonder: ***"Why don't I just evaluate $\sqrt{e}$ on my calculator? What's the point of using a series!? It just makes the calculation more complicated!"*** One answer lies in the question of how your *calculator* does the evaluation: at some level it must use the same series. Since only arithmetic operations are implemented directly in computer hardware, every calculation needs to be reduced to an arithmetic one. That's exactly what the series above does. The engineers who designed your calculator will almost certainly have used Taylor and Maclaurin series extensively to implement the essential functions like $\sin x$, $\cos x$, e^x, $\ln x$, etc.

In principle, if you were stuck on a desert island and needed to evaluate $\sqrt{e}$ to build your life raft (or bridge) to civilization, then there would be nothing preventing you now from evaluating $\sqrt{e}$ with a stick in the sand, using the arithmetic algorithms you learned in grade school. This might sound silly, but actually in mathematical problem solving you are very often stuck on a metaphorical desert island, needing to adapt available tools (e.g. arithmetic) to do something complicated (e.g. evaluating scientific functions). Taylor, Maclaurin and other series are often used this way.

Example: Use a Taylor series to evaluate $\int_0^{0.2} \sin(x^2)\,dx$ accurate to 4 significal digits.

Solution: You won't be able to find an antiderivative here. You could use Simpson's rule to approximate the integral, but sometimes using a Taylor series is better. We have the following Taylor (Maclaurin) series for $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{58}$$

so that

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots \tag{59}$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \tag{60}$$

Now finding the antiderivative is easy:

$$\int_0^{0.2} \sin(x^2)\,dx = \int_0^{0.2} \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \right) dx \tag{61}$$

$$= \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots \right]_0^{0.2} \tag{62}$$

$$= \underbrace{\frac{(0.2)^3}{3}}_{0.002667} - \underbrace{\frac{(0.2)^7}{7\cdot 3!}}_{3\times 10^{-7}} + \underbrace{\frac{(0.2)^{11}}{11\cdot 5!}}_{2\times 10^{-11}} - \cdots \tag{63}$$

Evidently we only need to keep the first two terms, since subsequent terms have no effect the first 4 significant digits:

$$\int_0^{0.2} \sin(x^2)\,dx \approx \frac{(0.2)^3}{3} - \frac{(0.2)^7}{7\cdot 3!} \approx \boxed{0.002666}. \tag{64}$$

Exercises: Sec. 11.4; #37, 39, 41, 45, 47

The End.